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Extension of an all-Mach Roe scheme able to deal with low Mach acoustics to full Euler system.

Thomas Galié* Jonathan Jung†‡ Ibtissem Lannabi†‡ Vincent Perrier†‡

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Abstract

We propose to extend the fix of Roe's approximate Riemann solver developed for the Barotropic Euler equations in [2] to the full Euler equations. This scheme is built mainly to handle low Mach acoustic waves. Moreover, compared to pressure-centered type schemes, this numerical fix has the advantage of improving the numerical solution in the sense that the oscillating modes are reduced. The theoretical study is based on a two-time scales asymptotic analysis. It is proved that the Euler system equipped with a general equation of state is consistent with a first-order wave system in a low Mach number regime. Similar analysis is performed at the discrete level on the Roe scheme to derive the new fix. Numerical tests confirm the results obtained for the Barotropic case about the ability of this fix to deal with both steady and low Mach acoustic computations also in the case of full Euler equations.

Introduction

We consider the Euler system of equations

$$\partial_t \mathcal{U} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathcal{U}) = 0, \quad \text{where} \quad \mathcal{U} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho E \end{pmatrix}, \quad \mathbf{f}(\mathcal{U}) = \begin{pmatrix} \rho \mathbf{u}^T \\ \rho \mathbf{u} \otimes \mathbf{u} + pI \\ (\rho E + p)\mathbf{u}^T \end{pmatrix}. \tag{1}$$

The vector $\mathcal{U}$ refers to the conservative variables composed of the density, momentum and total energy, and $\mathbf{f}$ to the physical fluxes in dimension d. The total energy is defined as the sum of the internal energy e and the kinetic energy $\rho E = \rho e + \rho \frac{\|\mathbf{u}\|^2}{2}$ and the pressure p satisfies the general state law $p = p(\rho, \rho e)$. The differential of this function writes

$$\mathrm{d}p = \chi \mathrm{d}\rho + \kappa \mathrm{d}(\rho e). \tag{2}$$

With those notations, the square sound velocity of the fluid is

$$a^2 = \chi + \kappa h,$$

with $h = e + p/\rho$ the specific enthalpy. Under the assumption that $a^2 > 0$ for all (ρ, e), the Euler system (1) is well-known to be hyperbolic with eigenvalues given by $\lambda_\pm = \mathbf{u} \cdot \mathbf{n} \pm a$ with genuinely nonlinear characteristic fields and $\lambda = \mathbf{u} \cdot \mathbf{n}$ with linear characteristic fields.

Given one time scale t_0, one length scale x_0, one density scale ρ_0 and one pressure scale p_0, the following dimensionless variables may be defined

$$\tilde{t} = \frac{t}{t_0}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{x_0}, \quad \tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{p} = \frac{p}{p_0}. \tag{3}$$

It is natural to scale the velocity by $u_0 = x_0/t_0$, and the sound speed by $a_0^2 = a^2(\rho_0, p_0)$. If the corresponding dimensionless variables are used instead of the original ones, the following system is obtained

$$\partial_{\tilde{t}} \tilde{\mathcal{U}} + \nabla_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{f}}\left(\tilde{\mathcal{U}}\right) = 0, \quad \text{where} \quad \tilde{\mathcal{U}} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}\tilde{\mathbf{u}} \\ \tilde{\rho}\tilde{E} \end{pmatrix}, \quad \tilde{\mathbf{f}}\left(\tilde{\mathcal{U}}\right) = \begin{pmatrix} \tilde{\rho}\tilde{\mathbf{u}}^T \\ \tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} + \dfrac{1}{\gamma M^2}\tilde{p} I \\ (\tilde{\rho}\tilde{E} + \frac{1}{\gamma}\tilde{p})\tilde{\mathbf{u}}^T \end{pmatrix}, \tag{4}$$

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with $\tilde{\mathbf{u}} = \mathbf{u}/u_0$, $M = u_0/a_0$, and $\tilde{E} = E/a_0^2$. The coefficient γ is defined as $\gamma = \dfrac{\rho_0 u_0^2}{p_0 M^2} = \dfrac{\rho_0 a_0^2}{p_0}$. Moreover, we have $\tilde{e} = e/a_0^2$, $\tilde{a}^2 = \tilde{\chi} + \tilde{\kappa}\tilde{h}$ with $\tilde{h} = h/a_0^2 = \tilde{e} + \tilde{p}/(\gamma\tilde{\rho})$, $\tilde{\chi} = \chi/a_0^2 = \dfrac{1}{a_0^2}\dfrac{\partial p}{\partial \rho} = \dfrac{p_0}{\rho_0 a_0^2}\dfrac{\partial \tilde{p}}{\partial \tilde{\rho}} = \dfrac{1}{\gamma}\dfrac{\partial \tilde{p}}{\partial \tilde{\rho}}$ and $\tilde{\kappa} = \kappa = \dfrac{\partial p}{\partial(\rho e)} = \dfrac{p_0}{\rho_0 a_0^2}\dfrac{\partial \tilde{p}}{\partial(\tilde{\rho}\tilde{e})} = \dfrac{1}{\gamma}\dfrac{\partial \tilde{p}}{\partial(\tilde{\rho}\tilde{e})}$. Additionally, given the definition of the total enthalpy $H = E + p/\rho$, one can define its dimensionless expression as $\tilde{H} = H/a_0^2$, which leads to $\tilde{H} = \tilde{E} + \tilde{p}/(\gamma\tilde{\rho})$.

Since some flows experience both acoustic and compressible effects and others may contain regions of both low and high Mach numbers in the same domain, a numerical scheme which is able to deal with these kinds of flows and which is accurate at low Mach number is required. As it is explained in [24], collocated density based schemes are able to handle mixed Mach number flows. More precisely, we are interested in finite volume schemes with the Roe numerical flux.

However, these methods face the well-known accuracy problem in a low Mach number regime, previously characterized by the creation of spurious pressure modes and explained via a single-scale asymptotic analysis as the inability of some schemes to produce discrete solutions which match the convective asymptotic solutions of the Euler system [8, 6, 7, 19, 13, 18]. Afterwards, using a two-scale asymptotic analysis [16], many papers have discussed the convective-acoustic limit obtained for Roe-type schemes [2, 26, 15], which gave another view to the accuracy problem. Based on Schochet's result [21], an interesting way to explain the accuracy problem by a simple analysis of a linear wave system has been proposed [4, 11]. In [11], it was shown that a given Godunov-type scheme is low Mach number accurate as soon as the long time limit of the matching discretization of a wave system ensures a uniform pressure (equal to the pressure imposed at the boundary) and a divergence free velocity (equal to the divergence free component of the Hodge Helmholtz Decomposition of the initial velocity): this statement is valid under specific boundary conditions that ensure the existence of the long time limit at the continuous and discrete levels (see [10]).

To overcome the accuracy problem, several fixes have been put forward. Most of them are built by modifying the artificial diffusion to achieve accurate low Mach number results. We refer to [9] for a review on the impact of the artificial diffusion on the accuracy of a low Mach number scheme.

Some fixes may be classified into pressure-centred schemes [4, 20, 22]. They consist in reducing the velocity diffusion in the discretization of a wave system leading to a centered discretization of the pressure gradient. Indeed, the modification of the artificial diffusion is carried out on the wave system derived from the asymptotic analysis of Euler's equations at low Mach number. Unfortunately, pressure-centered schemes introduce other problems such as checkerboard modes [3]. In contrast with pressure-centered fixes which aim at reducing the velocity diffusion, other fixes not only reduce the velocity diffusion but also add a pressure diffusion leading to a decentered discretization of the pressure gradient [2, 12, 14]. Once again, the change of the artificial diffusion is done on a wave system.

Other schemes are based on the use of preconditioning techniques which are known to compute accurately incompressible flows [25]. However, these fixes damp the acoustic variations on a very fast time scale [2].

This paper aims at extending the LMAAP fix (Low Mach Acoustic Accuracy Preserving) developed in [2] to the full Euler system. The interest in investigating this extension is motivated by the success of this scheme in taking into account acoustic phenomena, giving the right order of accuracy with a second order scheme and reducing the oscillatory modes in the solution, although the scheme is not Galilean invariant and its expression is asymmetric.

In the present study, a two-time scales asymptotic analysis is performed for Euler equations equipped with a general equation of state. We are able to prove that the first order pressure and zeroth order velocity satisfy a wave system with non constant coefficients. However, for the Roe scheme, the challenge in considering a general equation of state lies in the choice of the averaged partial derivatives of the thermodynamic variables derived from the considered equation of state. As we face some difficulties in proving the consistency with a wave system for the Roe scheme, the study is restricted to an equation of state where the first-order term of the asymptotic expansion of the partial derivatives of the thermodynamic variables are supposed to be constant. Hence, under this assumption, the consistency with the wave system is verified. Finally, the extension of LMAAP fix is deduced by following the same steps done in [2].

The paper is organized as follows: firstly, section 1 briefly recalls the formulation of the Roe scheme and its dimensionless expression in the subsonic case. After this, in section 2, the convective-acoustic limit is derived using two-time scales analysis. The study is performed for the continuous Euler equations by considering a general equation of state. However, for the discrete case, the first-order partial derivatives are supposed to be constant. Afterwards, in section 3, the extension of the Roe scheme corrected using the LMAAP fix from the barotropic case to the full Euler system is briefly presented. Finally, section 4 concerns the numerical tests. The LMAAP fix is assessed on three different problems: a low Mach Riemann problem, a channel with a singular pressure loss and an interaction of a low Mach isentropic stationary vortex with a low Mach acoustic wave.

1 The finite volume Roe scheme for low Mach flows

1.1 The finite volume Roe scheme

We define the following Roe averages

$$\begin{cases} \rho_{ij} = (1-\zeta_{ij})\,\rho_i + \zeta_{ij}\rho_j, \\ \mathbf{u}_{ij} = \zeta_{ij}\mathbf{u}_i + (1-\zeta_{ij})\,\mathbf{u}_j, \\ H_{ij} = \zeta_{ij}H_i + (1-\zeta_{ij})\,H_j, \end{cases} \qquad \text{where} \quad \zeta_{ij} = \frac{\sqrt{\rho_i}}{\sqrt{\rho_i}+\sqrt{\rho_j}}. \tag{5}$$

Let us notice that we can also write $\rho_{ij} = \sqrt{\rho_i \rho_j}$. Regarding the average partial derivatives χ_{ij}, κ_{ij}, there is not a unique choice. However, they should satisfy the relation

$$\Delta p = \chi_{ij}\Delta\rho + \kappa_{ij}\Delta(\rho e), \tag{6}$$

where $\Delta(\cdot) = (\cdot)_j - (\cdot)_i$ is the jump of a variable between cells i and j. Here we set them as follows

$$(\kappa_{ij}, \chi_{ij}) = \begin{cases} \left(\zeta_{ij}\kappa_i + (1-\zeta_{ij})\,\kappa_j, \dfrac{\Delta p}{\Delta\rho} - \kappa_{ij}\dfrac{\Delta(\rho e)}{\Delta\rho}\right) & \text{if } \Delta\rho \neq 0, \\ \left(\dfrac{\Delta p}{\Delta(\rho e)}, \chi_i\right) & \text{if } \Delta\rho = 0,\ \Delta(\rho e) \neq 0, \\ (\kappa_i, \chi_i) & \text{otherwise.} \end{cases}$$

The average sound velocity is then given by $a_{ij}^2 = \chi_{ij} + \kappa_{ij}h_{ij}$, with $h_{ij} = H_{ij} - \|\mathbf{u}_{ij}\|^2/2$. Other choices are possible for the average partial derivatives such as in [23] but these should not impact significantly results (at least for low Mach numbers flow which are of interest here). The Roe scheme for the Euler equations is defined as

$$|\Omega_i|\,\mathbf{d}_t\mathcal{U}_i(t) + \sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}|\,\Phi_{ij}^{\text{Roe}} = 0, \tag{7}$$

with

$$\begin{aligned} \Phi_{ij}^{\text{Roe}} =& \frac{\mathbf{f}(\mathcal{U}_i)+\mathbf{f}(\mathcal{U}_j)}{2}\cdot\mathbf{n}_{ij} - \frac{1}{4}\left|\mathbf{u}_{ij}\cdot\mathbf{n}_{ij} - a_{ij}\right| \frac{\Delta p - \rho_{ij}a_{ij}\Delta(\mathbf{u}\cdot\mathbf{n}_{ij})}{a_{ij}^2}\begin{pmatrix} 1 \\ \mathbf{u}_{ij} - a_{ij}\mathbf{n}_{ij} \\ H_{ij} - (\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})a_{ij} \end{pmatrix} \\ & - \frac{1}{2}\left|\mathbf{u}_{ij}\cdot\mathbf{n}_{ij}\right|\left(\Delta\rho - \frac{\Delta p}{a_{ij}^2}\right)\begin{pmatrix} 1 \\ \mathbf{u}_{ij} \\ H_{ij} - a_{ij}^2/\kappa_{ij} \end{pmatrix} - \frac{1}{2}\left|\mathbf{u}_{ij}\cdot\mathbf{n}_{ij}\right|\rho_{ij}\begin{pmatrix} 0 \\ \Delta\mathbf{u}^{\perp}(\mathbf{n}_{ij}) \\ \mathbf{u}_{ij}\cdot\Delta\mathbf{u}^{\perp}(\mathbf{n}_{ij}) \end{pmatrix} \\ & - \frac{1}{4}\left|\mathbf{u}_{ij}\cdot\mathbf{n}_{ij} + a_{ij}\right| \frac{\Delta p + \rho_{ij}a_{ij}\Delta(\mathbf{u}\cdot\mathbf{n}_{ij})}{a_{ij}^2}\begin{pmatrix} 1 \\ \mathbf{u}_{ij} + a_{ij}\mathbf{n}_{ij} \\ H_{ij} + (\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})a_{ij} \end{pmatrix}, \end{aligned} \tag{8}$$

where $\mathbf{u}^{\perp}(\mathbf{n}) := \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}$, denotes the tangential component of $\mathbf{u}$ with respect to normal direction $\mathbf{n}$.

1.2 Dimensionless expression of the Roe scheme in the subsonic case

The scheme (8) can be adimensioned by considering the dimensionless variables (3) and

$$\tilde{a}_{ij} = \frac{a_{ij}}{a_0}, \quad \tilde{H}_{ij} = \frac{H_{ij}}{a_0^2}, \quad |\tilde{\Gamma}_{ij}| = \frac{|\Gamma_{ij}|}{|\Gamma_0|}, \quad |\tilde{\Omega}_i| = \frac{|\Omega_i|}{|\Omega_0|}, \tag{9}$$

where $|\Gamma_0|/|\Omega_0| = 1/x_0$. In the subsonic case, the absolute values around the sonic waves can be removed and the flux can be simplified as

$$\begin{aligned} \widetilde{\Phi}_{ij}^{\text{Roe}} =& \frac{\tilde{\mathbf{f}}(\tilde{\mathcal{U}}_i)+\tilde{\mathbf{f}}(\tilde{\mathcal{U}}_j)}{2}\cdot\mathbf{n}_{ij} - \frac{1}{2}\left|\tilde{\mathbf{u}}_{ij}\cdot\mathbf{n}_{ij}\right|\left(\Delta\tilde{\rho} - \frac{\Delta\tilde{p}}{\gamma\tilde{a}_{ij}^2}\right)\begin{pmatrix} 1 \\ \tilde{\mathbf{u}}_{ij} \\ \tilde{H}_{ij} - \tilde{a}_{ij}^2/\kappa_{ij} \end{pmatrix} \\ & - \frac{1}{2}\left|\tilde{\mathbf{u}}_{ij}\cdot\mathbf{n}_{ij}\right|\tilde{\rho}_{ij}\begin{pmatrix} 0 \\ \Delta\tilde{\mathbf{u}}^{\perp}(\mathbf{n}_{ij}) \\ M^2\tilde{\mathbf{u}}_{ij}\cdot\Delta\tilde{\mathbf{u}}^{\perp}(\mathbf{n}_{ij}) \end{pmatrix} - \frac{1}{2}\frac{M\tilde{\rho}_{ij}\,(\tilde{\mathbf{u}}_{ij}\cdot\mathbf{n}_{ij})\,\Delta(\tilde{\mathbf{u}}\cdot\mathbf{n}_{ij}) + \frac{1}{\gamma M}\Delta\tilde{p}}{\tilde{a}_{ij}}\begin{pmatrix} 1 \\ \tilde{\mathbf{u}}_{ij} \\ \tilde{H}_{ij} \end{pmatrix} \\ & - \frac{1}{2}\left(\frac{(\tilde{\mathbf{u}}_{ij}\cdot\mathbf{n}_{ij})\,\Delta\tilde{p}}{\gamma\tilde{a}_{ij}^2} + \tilde{\rho}_{ij}\Delta(\tilde{\mathbf{u}}\cdot\mathbf{n}_{ij})\right)\begin{pmatrix} 0 \\ \frac{1}{M}\tilde{a}_{ij}\mathbf{n}_{ij} \\ M(\tilde{\mathbf{u}}_{ij}\cdot\mathbf{n}_{ij})\tilde{a}_{ij} \end{pmatrix} \\ & - \frac{\tilde{\theta}_{ij}-1}{2}\left(\tilde{\rho}_{ij}\Delta(\tilde{\mathbf{u}}\cdot\mathbf{n}_{ij})\right)\begin{pmatrix} 0 \\ \frac{1}{M}\tilde{a}_{ij}\mathbf{n}_{ij} \\ 0 \end{pmatrix}, \end{aligned} \tag{10}$$

where $\tilde{\theta}_{ij} = 1$ for the Roe scheme and $\tilde{\theta}_{ij} = \min(M_{ij}, 1)$ for the Dellacherie et al. fix with $M_{ij} = \max(M_i, M_j) = \max\left(\|\mathbf{u}_i\|/a_i, \|\mathbf{u}_j\|/a_j\right) = M\max\left(\|\tilde{\mathbf{u}}_i\|/\tilde{a}_i, \|\tilde{\mathbf{u}}_j\|/\tilde{a}_j\right)$. For more details, we refer to [5]. The system (10) will be used for performing the asymptotic analysis of the scheme.

2 Two-scale asymptotic analysis: acoustic limit on an incompressible flow

2.1 Behavior of the continuous equations in the low Mach number limit

In this section, we are interested in deriving a system which allows to separate the material time scale information (moving at velocity $u = \mathbf{u}\cdot\mathbf{n}$) from the acoustic time scale which corresponds to phenomena that move approximately at the velocity of the sound a_0 when the Mach number is sufficiently low. It means that we have now two reference time scales $T_{mat} = x_0/u_0 = t_0$ and $T_{ac} = x_0/a_0$ yielding two dimensionless time scales, namely

$$\tilde{t} = \frac{t}{T_{mat}} \quad \text{and} \quad \tau = \frac{t}{T_{ac}} = \frac{t}{x_0/a_0} = \frac{t}{u_0 \times T_{mat}/a_0} = \frac{t}{M \times T_{mat}} = \frac{\tilde{t}}{M}. \tag{11}$$

The material time $\tilde{t}$ and the acoustic time τ have a ratio of order M. This justifies to split the time dependency of the variables into two times: $\tilde{t}$ but also τ. Details on this development can be found in [16]. Then $\varphi \in \{\rho, \mathbf{u}, E\}$ can be written as an expansion in powers of the Mach number M

$$\tilde{\varphi}\left(\tilde{\mathbf{x}}, \tilde{t}, M\right) = \sum_{n=0}^{N} M^n \tilde{\varphi}^{(n)}\left(\tilde{\mathbf{x}}, \tilde{t}, \tau\right) + \mathcal{O}\left(M^{n+1}\right), \tag{12}$$

with $\tau = \tilde{t}/M$. Then the derivative with respect to the time is

$$\partial_{\tilde{t}}\tilde{\varphi}(\tilde{\mathbf{x}}, \tilde{t}, M) = \sum_{n=0}^{N} M^n \left(\partial_{\tilde{t}}\tilde{\varphi}^{(n)}\left(\tilde{\mathbf{x}}, \tilde{t}, \tau\right) + \frac{1}{M}\partial_{\tau}\tilde{\varphi}^{(n)}\left(\tilde{\mathbf{x}}, \tilde{t}, \tau\right)\right) + \mathcal{O}\left(M^{n+1}\right). \tag{13}$$

Let first note that since

$$\tilde{\rho}\tilde{E} = \tilde{\rho}\tilde{e} + M^2\tilde{\rho}\frac{\|\tilde{\mathbf{u}}\|^2}{2},$$

then we can deduce that

$$\tilde{\rho}\tilde{e} = (\tilde{\rho}\tilde{E})^{(0)} + M(\tilde{\rho}\tilde{E})^{(1)} + M^2\left((\tilde{\rho}\tilde{E})^{(2)} - \tilde{\rho}^{(0)}\frac{\left\|\tilde{\mathbf{u}}^{(0)}\right\|^2}{2}\right) + \mathcal{O}\left(M^3\right),$$

which naturally gives a definition to $(\tilde{\rho}\tilde{e})^{(i)},\ i = 0, \ldots, 2$ in the asymptotic expansion:

$$(\tilde{\rho}\tilde{e})^{(0)} = (\tilde{\rho}\tilde{E})^{(0)}, \quad (\tilde{\rho}\tilde{e})^{(1)} = (\tilde{\rho}\tilde{E})^{(1)}, \quad (\tilde{\rho}\tilde{e})^{(2)} = (\tilde{\rho}\tilde{E})^{(2)} - \tilde{\rho}^{(0)}\frac{\left\|\tilde{\mathbf{u}}^{(0)}\right\|^2}{2}.$$

Concerning the asymptotic development on the pressure, a first order development gives

$$\begin{aligned}
\tilde{p} = \tilde{p}(\tilde{\rho}, \tilde{\rho}\tilde{e}) &= \tilde{p}\left(\tilde{\rho}^{(0)} + M\tilde{\rho}^{(1)} + \mathcal{O}\left(M^2\right), (\tilde{\rho}\tilde{e})^{(0)} + M(\tilde{\rho}\tilde{e})^{(1)} + \mathcal{O}\left(M^2\right)\right) \\
&= \tilde{p}\left(\tilde{\rho}^{(0)}, (\tilde{\rho}\tilde{e})^{(0)}\right) + \frac{\partial\tilde{p}^{(0)}}{\partial\tilde{\rho}^{(0)}}M\tilde{\rho}^{(1)} + \frac{\partial\tilde{p}^{(0)}}{\partial(\tilde{\rho}\tilde{e})^{(0)}}M(\tilde{\rho}\tilde{e})^{(1)} + \mathcal{O}\left(M^2\right) \\
&= \tilde{p}^{(0)} + M\tilde{p}^{(1)} + \mathcal{O}\left(M^2\right),
\end{aligned} \tag{14}$$

with

$$\frac{\partial\tilde{p}^{(0)}}{\partial\tilde{\rho}^{(0)}} = \gamma\tilde{\chi}^{(0)}, \qquad \frac{\partial\tilde{p}^{(0)}}{\partial(\tilde{\rho}\tilde{e})^{(0)}} = \gamma\tilde{\kappa}^{(0)}, \qquad \tilde{p}^{(0)} = \tilde{p}\left(\tilde{\rho}^{(0)}, (\tilde{\rho}\tilde{e})^{(0)}\right), \qquad \tilde{p}^{(1)} = \gamma\tilde{\chi}^{(0)}\tilde{\rho}^{(1)} + \gamma\tilde{\kappa}^{(0)}(\tilde{\rho}\tilde{e})^{(1)}.$$

Applying a differential operator d on (14) thus leads to

$$\mathrm{d}\tilde{p} = \mathrm{d}\tilde{p}^{(0)} + M\mathrm{d}\tilde{p}^{(1)} + \mathcal{O}\left(M^2\right). \tag{15}$$

with

$$\mathrm{d}\tilde{p}^{(0)} = \gamma\tilde{\chi}^{(0)}\mathrm{d}\tilde{\rho}^{(0)} + \gamma\tilde{\kappa}^{(0)}\mathrm{d}\left(\tilde{\rho}\tilde{e}\right)^{(0)}, \tag{16}$$

$$\mathrm{d}\tilde{p}^{(1)} = \mathrm{d}\left(\gamma\tilde{\chi}^{(0)}\tilde{\rho}^{(1)}\right) + \mathrm{d}\left(\gamma\tilde{\kappa}^{(0)}\left(\tilde{\rho}\tilde{e}\right)^{(1)}\right). \tag{17}$$

Proposition 1. *(Properties of the thermodynamic variables) The pressure* $\tilde{p}^{(0)}$ *is such that*

$$\tilde{p}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t},\tau\right)=\tilde{p}^{(0)}(\tilde{t}), \tag{18}$$

and $\tilde{\rho}^{(0)}$, $\tilde{E}^{(0)}$, $\tilde{\chi}^{(0)}$, $\tilde{\kappa}^{(0)}$ *do not depend on* τ.

Proof. By injecting the development (13) in (4), the momentum equation at order M^{-2} gives

$$\nabla_{\tilde{\mathbf{x}}}\tilde{p}^{(0)}=0, \tag{19}$$

At order M^{-1}, we obtain

$$\begin{cases}\partial_\tau\tilde{\rho}^{(0)}=0, & \text{(20a)}\\ \partial_\tau\left(\tilde{\rho}^{(0)}\tilde{\mathbf{u}}^{(0)}\right)+\dfrac{1}{\gamma}\nabla_{\tilde{\mathbf{x}}}\tilde{p}^{(1)}=0, & \text{(20b)}\\ \partial_\tau\left(\tilde{\rho}^{(0)}\tilde{E}^{(0)}\right)=0. & \text{(20c)}\end{cases}$$

Since we have (16), multiplying (20a) by $\gamma\tilde{\chi}^{(0)}$ and (20c) by $\gamma\tilde{\kappa}^{(0)}$ and using the formula $(\tilde{\rho}\tilde{e})^{(0)}=(\tilde{\rho}\tilde{E})^{(0)}$ lead to $\partial_\tau\tilde{p}^{(0)}=0$ and thus independence of $\tilde{p}^{(0)}$ from $\tilde{\mathbf{x}}$ and τ. Note that, from (20a) and (20c), we obtain

$$\begin{aligned}\tilde{\rho}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t},\tau\right)&=\tilde{\rho}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t}\right),\\ \tilde{\rho}^{(0)}\tilde{E}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t},\tau\right)&=\tilde{\rho}^{(0)}\tilde{E}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t}\right),\end{aligned} \tag{21}$$

and thus

$$\tilde{\chi}^{(0)}=\frac{1}{\gamma}\frac{\partial\tilde{p}^{(0)}}{\partial\tilde{\rho}^{(0)}}=\tilde{\chi}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t}\right),\quad \tilde{\kappa}^{(0)}=\frac{1}{\gamma}\frac{\partial\tilde{p}^{(0)}}{\partial(\tilde{\rho}\tilde{e})^{(0)}}=\tilde{\kappa}^{(0)}\left(\tilde{\mathbf{x}},\tilde{t}\right).$$

□

Proposition 2. *(Link with the wave system)* $\left(\tilde{p}^{(1)},\tilde{\mathbf{u}}^{(0)}\right)$ *satisfies the wave system*

$$\begin{cases}\partial_\tau\tilde{p}^{(1)}+\gamma\tilde{\rho}^{(0)}\left(\tilde{a}^{(0)}\right)^2\nabla_{\tilde{\mathbf{x}}}\cdot\tilde{\mathbf{u}}^{(0)}=-\mathrm{d}_{\tilde{t}}\tilde{p}^{(0)}, & \text{(22a)}\\ \partial_\tau\tilde{\mathbf{u}}^{(0)}+\dfrac{1}{\gamma\tilde{\rho}^{(0)}}\nabla_{\tilde{\mathbf{x}}}\tilde{p}^{(1)}=0, & \text{(22b)}\end{cases}$$

where $\mathrm{d}_{\tilde{t}}$ *denotes the total derivative.*

Proof. From the insertion of the development (13) in (4), we get at order M^0

$$\begin{cases}\partial_{\tilde{t}}\tilde{\rho}^{(0)}+\partial_\tau\tilde{\rho}^{(1)}+\nabla_{\tilde{\mathbf{x}}}\cdot\left(\tilde{\rho}^{(0)}\tilde{\mathbf{u}}^{(0)}\right)=0, & \text{(23a)}\\ \partial_{\tilde{t}}(\tilde{\rho}^{(0)}\tilde{E}^{(0)})+\partial_\tau\left(\tilde{\rho}\tilde{E}\right)^{(1)}+\nabla_{\tilde{\mathbf{x}}}\cdot(\tilde{\rho}^{(0)}\tilde{E}^{(0)}\tilde{\mathbf{u}}^{(0)}+\dfrac{1}{\gamma}\tilde{p}^{(0)}\tilde{\mathbf{u}}^{(0)})=0. & \text{(23b)}\end{cases}$$

Applying the partial derivative ∂_τ as the differential operator in (17) on $\tilde{p}^{(1)}$ leads to

$$\partial_\tau\tilde{p}^{(1)}=\gamma\tilde{\chi}^{(0)}\partial_\tau\tilde{\rho}^{(1)}+\gamma\tilde{\kappa}^{(0)}\partial_\tau\left(\tilde{\rho}\tilde{e}\right)^{(1)}, \tag{24}$$

since $\tilde{\chi}^{(0)}$ and $\tilde{\kappa}^{(0)}$ do not depend on τ. Since $\tilde{p}^{(0)}$ does not depend on $\tilde{\mathbf{x}}$, with (16) we get

$$\begin{aligned}\gamma\tilde{\chi}^{(0)}\nabla_{\tilde{\mathbf{x}}}\tilde{\rho}^{(0)}+\gamma\tilde{\kappa}^{(0)}\nabla_{\tilde{\mathbf{x}}}\left(\tilde{\rho}\tilde{H}\right)^{(0)}&=\gamma\tilde{\chi}^{(0)}\nabla_{\tilde{\mathbf{x}}}\tilde{\rho}^{(0)}+\gamma\tilde{\kappa}^{(0)}\nabla_{\tilde{\mathbf{x}}}\left(\tilde{\rho}\tilde{E}\right)^{(0)}+\tilde{\kappa}^{(0)}\nabla_{\tilde{\mathbf{x}}}\tilde{p}^{(0)}\\ &=\gamma\tilde{\chi}^{(0)}\nabla_{\tilde{\mathbf{x}}}\tilde{\rho}^{(0)}+\gamma\tilde{\kappa}^{(0)}\nabla_{\tilde{\mathbf{x}}}\left(\tilde{\rho}\tilde{e}\right)^{(0)}=0.\end{aligned}$$

In addition, we have

$$\begin{aligned}\gamma\tilde{\chi}^{(0)}\tilde{\rho}^{(0)}+\gamma\tilde{\kappa}^{(0)}\tilde{\rho}^{(0)}\tilde{E}^{(0)}+\tilde{\kappa}^{(0)}\tilde{p}^{(0)}&=\gamma\tilde{\rho}^{(0)}\left(\tilde{\chi}^{(0)}+\tilde{\kappa}^{(0)}\tilde{e}^{(0)}+\tilde{\kappa}^{(0)}\tilde{p}^{(0)}/(\gamma\tilde{\rho}^{(0)})\right)\\ &=\gamma\tilde{\rho}^{(0)}\left(\tilde{\chi}^{(0)}+\tilde{\kappa}^{(0)}\tilde{h}^{(0)}\right)\\ &=\gamma\tilde{\rho}^{(0)}\left(\tilde{a}^{(0)}\right)^2.\end{aligned}$$

Then, by multiplying (23a) and (23b) respectively by $\gamma\tilde{\chi}^{(0)}$ and $\gamma\tilde{\kappa}^{(0)}$ and by summing the two, we obtain (22a). (22b) is obtained using (20b) and (20a). □

2.2 Behavior of the Roe scheme and the Roe scheme corrected with the Dellacherie et al. or Rieper fixes

The study of the semi-discrete scheme (10) is organized by following the same steps as in subsection 2.1 for the continuous case. However, we face an obstacle for proving uniformity of $\tilde{p}^{(0)}$ and deriving the consistency with the wave system whenever $\tilde{\chi}^{(0)}$ and $\tilde{\kappa}^{(0)}$ are not constant. For this sake, the study is limited to an equation of state where the first order term of the pressure p in the Mach number expansion is a linear function of ρ and ρe.

Hypothesis 1. *We suppose in what follows that χ and κ are constant in space, i.e. $\tilde{\chi}_{ij}^{(0)} = \tilde{\chi}_i^{(0)} = \tilde{\chi}^{(0)}, \quad \tilde{\kappa}_{ij}^{(0)} = \tilde{\kappa}_i^{(0)} = \tilde{\kappa}^{(0)}, \ \forall i,j.$*

The following proposition is the discrete version of Proposition 1.

Proposition 3. *(Properties of the thermodynamic variables)*
If $\tilde{p}^{(0)}(\tilde{\mathbf{x}},\tilde{t},\tau=0) = \tilde{p}_0^{(0)}(\tilde{t})$ and $\tilde{p}^{(0)}$ is also equal to $\tilde{p}_0^{(0)}$ at the boundaries, then

$$\forall \tilde{t}>0, \ \forall \tau \geq 0, \ \forall i, \quad \tilde{p}_i^{(0)}(\tilde{t},\tau) = \tilde{p}_0^{(0)}(\tilde{t}).$$

Moreover, we have

$$\forall \tilde{t}>0, \ \forall \tau \geq 0, \ \forall i, \quad \tilde{\rho}_i^{(0)}(\tilde{t},\tau) = \tilde{\rho}_i^{(0)}(\tilde{t}), \quad \tilde{E}_i^{(0)}(\tilde{t},\tau) = \tilde{E}_i^{(0)}(\tilde{t}),$$

$$\forall \tilde{t}>0, \ \forall \tau \geq 0, \ \forall i, \quad \tilde{\rho}_i^{(0)} \left(\tilde{a}_i^{(0)}\right)^2 (\tilde{t},\tau) = \tilde{\rho}^{(0)} \left(\tilde{a}^{(0)}\right)^2 (\tilde{t}).$$

Proof. By injecting (12) in (7)-(10), we obtain at order M^{-2}

$$\sum_{\Gamma_{ij}\subset\partial\Omega_i} \left|\tilde{\Gamma}_{ij}\right| \frac{\tilde{p}_i^{(0)} + \tilde{p}_j^{(0)}}{2} \mathbf{n}_{ij} = 0, \tag{25}$$

and at order M^{-1}

$$\begin{cases} \partial_\tau \tilde{\rho}_i^{(0)} + \dfrac{1}{|\tilde{\Omega}_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} \left|\tilde{\Gamma}_{ij}\right| \frac{1}{2\gamma \tilde{a}_{ij}^{(0)}} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)}\right) = 0, & \text{(26a)} \\ \partial_\tau \left(\tilde{\rho}_i \tilde{\mathbf{u}}_i\right)^{(0)} + \dfrac{1}{|\tilde{\Omega}_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} \left|\tilde{\Gamma}_{ij}\right| \left[\frac{1}{2\gamma \tilde{a}_{ij}^{(0)}} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)}\right) \left(\tilde{\mathbf{u}}_{ij}^{(0)} + \left(\tilde{\mathbf{u}}_{ij}^{(0)} \cdot \mathbf{n}_{ij}\right) \mathbf{n}_{ij}\right) \right. \\ \qquad \left. + \left(\dfrac{\tilde{p}_i^{(1)} + \tilde{p}_j^{(1)}}{2\gamma} + \dfrac{\delta \tilde{\rho}_{ij}^{(0)} \tilde{a}_{ij}^{(0)}}{2} (\tilde{\mathbf{u}}_i^{(0)} - \tilde{\mathbf{u}}_j^{(0)}) \cdot \mathbf{n}_{ij} \right) \mathbf{n}_{ij} \right] = 0, & \text{(26b)} \\ \partial_\tau \left(\tilde{\rho}_i \tilde{E}_i\right)^{(0)} + \dfrac{1}{|\tilde{\Omega}_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} \left|\tilde{\Gamma}_{ij}\right| \frac{\tilde{H}_{ij}^{(0)}}{2\gamma \tilde{a}_{ij}^{(0)}} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)}\right) = 0, & \text{(26c)} \end{cases}$$

where $\delta = 1$ for the Roe scheme and $\delta = 0$ for the Dellacherie et al. or Rieper fixes.

By multiplying (26a) by $\gamma\tilde{\chi}^{(0)}$ and (26c) by $\gamma\tilde{\kappa}^{(0)}$, that are supposed to be constant, and summing the resulting equations, we get

$$\gamma\tilde{\chi}^{(0)} \partial_\tau \tilde{\rho}_i^{(0)} + \gamma\tilde{\kappa}^{(0)} \partial_\tau \left(\tilde{\rho}_i \tilde{E}_i\right)^{(0)} + \frac{1}{|\tilde{\Omega}_i|} \sum_{\Gamma_{ij}\subset\partial\Omega_i} \left|\tilde{\Gamma}_{ij}\right| \frac{\tilde{\chi}^{(0)} + \tilde{\kappa}^{(0)} \tilde{H}_{ij}^{(0)}}{2\tilde{a}_{ij}^{(0)}} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)}\right) = 0.$$

Since $\left(\tilde{\rho}_i \tilde{E}_i\right)^{(0)} = \left(\tilde{\rho}_i \tilde{e}_i\right)^{(0)}$ (let us remind that indeed $\tilde{\rho}_i \tilde{E}_i = \tilde{\rho}_i \tilde{e}_i + \mathcal{O}(M^2)$), the above equation is thus simplified into

$$\partial_\tau \tilde{p}_i^{(0)} + \frac{1}{|\tilde{\Omega}_i|} \sum_{\Gamma_{ij}\subset\partial\Omega_i} |\tilde{\Gamma}_{ij}| \frac{\tilde{a}_{ij}^{(0)}}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)}\right) = 0. \tag{27}$$

By multiplying (27) by $|\tilde{\Omega}_i| \left(\tilde{p}_i^{(0)} - \tilde{p}_0^{(0)}\right)$ and summing over all cells Ω_i, we get

$$\partial_\tau \left(\sum_i \frac{|\tilde{\Omega}_i|}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_0^{(0)}\right)^2 \right) + \sum_i \sum_{\Gamma_{ij}\subset\partial\Omega_i} |\tilde{\Gamma}_{ij}| \frac{\tilde{a}_{ij}^{(0)}}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)}\right) \left(\tilde{p}_i^{(0)} - \tilde{p}_0^{(0)}\right) = 0,$$

rearranging the second sum as a sum over the edges and distinguishing inner and boundary edges yield

$$\partial_\tau \left(\sum_i \frac{|\tilde{\Omega}_i|}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_0^{(0)} \right)^2 \right) = - \sum_{\Gamma_{ij} \subset \mathring{\Omega}} |\tilde{\Gamma}_{ij}| \frac{\tilde{a}_{ij}^{(0)}}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_j^{(0)} \right)^2 - \sum_{\Gamma_{ij} \subset \partial\Omega} |\tilde{\Gamma}_{ij}| \frac{\tilde{a}_{ij}^{(0)}}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_0^{(0)} \right)^2 \leq 0.$$

where $\mathring{\Omega} = \Omega \setminus \partial\Omega$ is the interior of Ω. Note that we used that in the boundaries $\partial\Omega$, $\tilde{p}_j^{(0)} = \tilde{p}_0^{(0)}$. Then the positive function $\tau \mapsto \sum_i \frac{|\tilde{\Omega}_i|}{2} \left(\tilde{p}_i^{(0)} - \tilde{p}_0^{(0)} \right)^2$ decreases. Since at $\tau = 0$, $\tilde{p}^{(0)}$ is uniform (equal to $\tilde{p}_0^{(0)}$), this function vanishes at $\tau = 0$ and then at any $\tau > 0$. This concludes the proof for $\tilde{p}^{(0)}$. The properties on $\tilde{\rho}_i^{(0)}$ and $\tilde{E}_i^{(0)}$ are directly deduced from (26a) and (26c). Which yields independence of $\tilde{\rho}^{(0)} \left(\tilde{a}^{(0)} \right)^2$ with respect to τ. Moreover, using the formula (6) and the uniformity of $\tilde{p}^{(0)}$, we have

$$\Delta \left(\tilde{\rho}^{(0)} \left(\tilde{a}^{(0)} \right)^2 \right) = \Delta \left(\tilde{\rho}^{(0)} \left(\tilde{\chi}^{(0)} + \tilde{\kappa}^{(0)} \tilde{H}^{(0)} \right) \right) = \tilde{\chi}^{(0)} \Delta \tilde{\rho}^{(0)} + \tilde{\kappa}^{(0)} \Delta \left(\tilde{\rho} \tilde{E} \right)^{(0)} + \frac{\tilde{\kappa}^{(0)}}{\gamma} \Delta \tilde{p}^{(0)} = 0,$$

which justifies the uniformity of $\tilde{\rho}^{(0)} \left(\tilde{a}^{(0)} \right)^2$. Note that the latter is uniform also at the continuous level, when $\chi^{(0)}$ and $\kappa^{(0)}$ are constant.

□

Still following the steps of (2.1), the following proposition gives the equivalent of Proposition 4 for the semi-discrete scheme.

Proposition 4. *(Consistency with the wave equation) Under the hypothesis of Proposition 3, $\left(\tilde{p}^{(1)}, \tilde{\mathbf{u}}^{(0)} \right)$ satisfies the semi-discrete (discrete in space and continuous in time) wave equation*

$$\begin{cases} \partial_\tau \tilde{p}_i^{(1)} + \dfrac{1}{|\tilde{\Omega}_i|} \displaystyle\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\tilde{\Gamma}_{ij}| \left(\gamma \tilde{\rho}^{(0)} \left(\tilde{a}^{(0)} \right)^2 \frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2} \cdot \mathbf{n}_{ij} + \frac{\tilde{a}_{ij}^{(0)}}{2} \left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)} \right) \right) = -\mathrm{d}_{\tilde{t}} \tilde{p}_0^{(0)}, \\ \partial_\tau \tilde{\mathbf{u}}_i^{(0)} + \dfrac{1}{\tilde{\rho}_i^{(0)}} \dfrac{1}{|\tilde{\Omega}_i|} \displaystyle\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\tilde{\Gamma}_{ij}| \left(\frac{\tilde{p}_i^{(1)} + \tilde{p}_j^{(1)}}{2\gamma} \mathbf{n}_{ij} + \frac{\delta \tilde{\rho}_{ij}^{(0)} \tilde{a}_{ij}^{(0)}}{2} \left(\left(\tilde{\mathbf{u}}_i^{(0)} - \tilde{\mathbf{u}}_j^{(0)} \right) \cdot \mathbf{n}_{ij} \right) \mathbf{n}_{ij} \right) = 0, \end{cases} \tag{28}$$

where $\delta = 1$ for the Roe scheme and $\delta = 0$ for the Dellacherie et al. or Rieper fixes.

Proof. Under the hypothesis of the Proposition 3, the equation on $\tilde{\mathbf{u}}^{(0)}$ is directly deduced from (26b) and the result of Proposition 3.

From the injection of (12) in (7)-(10), we have at order M^0

$$\begin{aligned} &\partial_{\tilde{t}} \tilde{\rho}_i^{(0)} + \partial_\tau \tilde{\rho}_i^{(1)} + \frac{1}{|\tilde{\Omega}_i|} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \left| \tilde{\Gamma}_{ij} \right| \left[\frac{\tilde{\rho}_i^{(0)} \tilde{\mathbf{u}}_i^{(0)} + \tilde{\rho}_j^{(0)} \tilde{\mathbf{u}}_j^{(0)}}{2} \cdot \mathbf{n}_{ij} \right. \\ &\qquad \left. + \frac{1}{2} \left| \tilde{\mathbf{u}}_{ij}^{(0)} \cdot \mathbf{n}_{ij} \right| \left(\tilde{\rho}_i^{(0)} - \tilde{\rho}_j^{(0)} \right) + \frac{1}{2\gamma \tilde{a}_{ij}^{(0)}} \left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)} \right) \right] = 0 \end{aligned} \tag{29a}$$

$$\begin{aligned} &\partial_{\tilde{t}} \left(\tilde{\rho}_i \tilde{E}_i \right)^{(0)} + \partial_\tau \left(\tilde{\rho}_i \tilde{E}_i \right)^{(1)} + \frac{1}{|\tilde{\Omega}_i|} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \left| \tilde{\Gamma}_{ij} \right| \left[\frac{\tilde{\rho}_i^{(0)} \tilde{E}_i^{(0)} \tilde{\mathbf{u}}_i^{(0)} + \tilde{\rho}_j^{(0)} \tilde{E}_j^{(0)} \tilde{\mathbf{u}}_j^{(0)}}{2} \cdot \mathbf{n}_{ij} \right. \\ &\qquad + \tilde{p}^{(0)} \left(\frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2\gamma} \right) \cdot \mathbf{n}_{ij} + \frac{1}{2} \left| \tilde{\mathbf{u}}_{ij}^{(0)} \cdot \mathbf{n}_{ij} \right| \left(\tilde{\rho}_i^{(0)} - \tilde{\rho}_j^{(0)} \right) \left(\tilde{H}_{ij}^{(0)} - \frac{\left(\tilde{a}_{ij}^{(0)} \right)^2}{\tilde{\kappa}^{(0)}} \right) \\ &\qquad \left. + \frac{\tilde{H}_{ij}^{(0)}}{2\gamma \tilde{a}_{ij}^{(0)}} \left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)} \right) \right] = 0. \end{aligned} \tag{29b}$$

By multiplying (29a) by $\gamma \tilde{\chi}^{(0)}$ and (29b) by $\gamma \tilde{\kappa}^{(0)}$, then summing the resulting equations, we get

$$\begin{aligned} \partial_\tau \tilde{p}_i^{(1)} + \frac{1}{|\tilde{\Omega}_i|} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \left| \tilde{\Gamma}_{ij} \right| \left[\gamma \tilde{\chi}^{(0)} \left(\frac{\tilde{\rho}_i^{(0)} \tilde{\mathbf{u}}_i^{(0)} + \tilde{\rho}_j^{(0)} \tilde{\mathbf{u}}_j^{(0)}}{2} \right) \cdot \mathbf{n}_{ij} + \gamma \tilde{\kappa}^{(0)} \left(\frac{\tilde{\rho}_i^{(0)} \tilde{E}_i^{(0)} \tilde{\mathbf{u}}_i^{(0)} + \tilde{\rho}_j^{(0)} \tilde{E}_j^{(0)} \tilde{\mathbf{u}}_j^{(0)}}{2} \right) \cdot \mathbf{n}_{ij} \right. \\ \left. + \gamma \tilde{\kappa}^{(0)} \tilde{p}^{(0)} \left(\frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2\gamma} \right) \cdot \mathbf{n}_{ij} + \frac{\tilde{a}_{ij}^{(0)}}{2} \left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)} \right) \right] = -\mathrm{d}_{\tilde{t}} \tilde{p}_0^{(0)}, \end{aligned}$$

because, since $\tilde{\chi}^{(0)}$ and $\tilde{\kappa}^{(0)}$ have been supposed to be constant in space, we have $\left(\tilde{a}_{ij}^{(0)}\right)^2 = \tilde{\chi}^{(0)} + \tilde{\kappa}^{(0)}\tilde{H}_{ij}^{(0)}$. We will now use the following equality

$$\frac{\tilde{\varphi}_i^{(0)}\tilde{\mathbf{u}}_i^{(0)} + \tilde{\varphi}_j^{(0)}\tilde{\mathbf{u}}_j^{(0)}}{2} = \tilde{\varphi}_i^{(0)}\left(\frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2}\right) - \tilde{\mathbf{u}}_j^{(0)}\left(\frac{\tilde{\varphi}_i^{(0)} - \tilde{\varphi}_j^{(0)}}{2}\right) = \tilde{\varphi}_i^{(0)}\left(\frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2}\right) + \frac{\tilde{\mathbf{u}}_j^{(0)}}{2}\Delta\tilde{\varphi}^{(0)},$$

for $\varphi \in \{\rho, \rho E\}$. We obtain

$$\partial_\tau \tilde{p}_i^{(1)} + \frac{1}{|\tilde{\Omega}_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}\left|\tilde{\Gamma}_{ij}\right|\left[\gamma\tilde{\rho}_i^{(0)}\left(\tilde{\chi}^{(0)} + \tilde{\kappa}^{(0)}\tilde{E}_i^{(0)} + \tilde{\kappa}^{(0)}\tilde{p}^{(0)}/\left(\gamma\tilde{\rho}_i^{(0)}\right)\right)\left(\frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2}\right)\cdot\mathbf{n}_{ij}\right.$$
$$\left. + \frac{\tilde{\mathbf{u}}_j^{(0)}\cdot\mathbf{n}_{ij}}{2}\gamma\left(\tilde{\chi}^{(0)}\Delta\tilde{\rho}^{(0)} + \tilde{\kappa}^{(0)}\Delta\left(\tilde{\rho}\tilde{E}\right)^{(0)}\right) + \frac{\tilde{a}_{ij}^{(0)}}{2}\left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)}\right)\right] = -\mathrm{d}_{\tilde{t}}\tilde{p}_0^{(0)}.$$

Since $\tilde{\chi}^{(0)}\Delta\tilde{\rho}^{(0)} + \tilde{\kappa}^{(0)}\Delta\left(\tilde{\rho}\tilde{E}\right)^{(0)} = \Delta\tilde{p}^{(0)} = 0$, $\tilde{\chi}^{(0)} + \tilde{\kappa}^{(0)}\tilde{E}_i^{(0)} + \tilde{\kappa}^{(0)}\tilde{p}^{(0)}/\left(\gamma\tilde{\rho}_i^{(0)}\right) = \left(\tilde{a}_i^{(0)}\right)^2$ and $\tilde{\rho}^{(0)}\left(\tilde{a}^{(0)}\right)^2$ is uniform, this concludes the proof. □

3 Extension of the LMAAP scheme: accurate in the low Mach number limit for steady and for acoustics computations

In this section, we aim to extend the LMAAP scheme developed in [2] for the Barotropic Euler equations to the full Euler system (1). We first present a general expression of the new scheme. Next, conditions for ensuring a semi-discrete energy inequality are found. Then, the expression of the scheme is deduced for the wave system obtained in Proposition 2 and eventually to the full Euler system.

3.1 Extension of the scheme for the wave system

Supposing that the domain is infinite or periodic, hypothesis of Proposition 3 holds, $\mathrm{d}_{\tilde{t}}\tilde{p}_0^{(0)}(\tilde{t}) = 0$ for any $\tilde{t}$ and that Hypothesis 1 holds, the expression of the scheme obtained in Proposition 4 is generalized to

$$\begin{cases} \partial_\tau \tilde{p}_i^{(1)} + \dfrac{1}{|\tilde{\Omega}_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\tilde{\Gamma}_{ij}|\left(\gamma\tilde{\rho}^{(0)}\left(\tilde{a}^{(0)}\right)^2\frac{\tilde{\mathbf{u}}_i^{(0)} + \tilde{\mathbf{u}}_j^{(0)}}{2}\cdot\mathbf{n}_{ij} + C_{ij}^{11}\left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)}\right) + \mathbf{C}_{ij}^{12}\cdot\left(\tilde{\mathbf{u}}_i^{(0)} - \tilde{\mathbf{u}}_j^{(0)}\right)\right) = 0, & \text{(30a)} \\ \partial_\tau \tilde{\mathbf{u}}_i^{(0)} + \dfrac{1}{|\tilde{\Omega}_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\tilde{\Gamma}_{ij}|\left(\frac{1}{\gamma\tilde{\rho}_i^{(0)}}\frac{\tilde{p}_i^{(1)} + \tilde{p}_j^{(1)}}{2}\mathbf{n}_{ij} + \mathbf{C}_{ij}^{21}\left(\tilde{p}_i^{(1)} - \tilde{p}_j^{(1)}\right) + C_{ij}^{22}\left(\left(\tilde{\mathbf{u}}_i^{(0)} - \tilde{\mathbf{u}}_j^{(0)}\right)\cdot\mathbf{n}_{ij}\right)\mathbf{n}_{ij}\right) = 0. & \text{(30b)} \end{cases}$$

We remark that the Roe scheme is consistent at low Mach number with a discretization of the wave system (30) with $C_{ij}^{11} = \tilde{a}_{ij}^{(0)}/2$, $\mathbf{C}_{ij}^{12} = \mathbf{C}_{ij}^{21} = \mathbf{0}$, $C_{ij}^{22} = \tilde{\rho}_{ij}^{(0)}\tilde{a}_{ij}^{(0)}/2\tilde{\rho}_i^{(0)}$ and the Roe scheme with Dellacherie or Rieper fixes correspond to a discretization of the wave system (30) with $C_{ij}^{11} = \tilde{a}_{ij}^{(0)}/2$, $\mathbf{C}_{ij}^{12} = \mathbf{C}_{ij}^{21} = \mathbf{0}$, $C_{ij}^{22} = 0$, which means that we have a centered discretization of the pressure gradient.

Proposition 5. *Under the Hypothesis 1, the assumptions of Proposition 3 and*

$$C_{ij}^{11} = C_{ji}^{11}, \quad \mathbf{C}_{ij}^{12} = \mathbf{C}_{ji}^{12}, \quad \tilde{\rho}_i^{(0)}\mathbf{C}_{ij}^{21} = \tilde{\rho}_j^{(0)}\mathbf{C}_{ji}^{21}, \quad \tilde{\rho}_i^{(0)}C_{ij}^{22} = \tilde{\rho}_j^{(0)}C_{ji}^{22}, \tag{31}$$

the scheme (30) satisfies the semi-discrete energy inequality

$$\partial_\tau\left(\sum_i|\tilde{\Omega}_i|\frac{\left(\tilde{p}_i^{(1)}\right)^2 + \left(\gamma\tilde{a}_i^{(0)}\tilde{\rho}_i^{(0)}\right)^2\|\tilde{\mathbf{u}}_i^{(0)}\|^2}{2}\right) \leq 0, \tag{32}$$

if and only if

$$C_{ij}^{11} \geq 0, \quad \mathbf{C}_{ij}^{12} + \left(\gamma^2\tilde{\rho}^{(0)}\left(\tilde{a}^{(0)}\right)^2\right)\tilde{\rho}_i^{(0)}\mathbf{C}_{ij}^{21} = 0, \quad C_{ij}^{22} \geq 0. \tag{33}$$

The proof is similar to the one done in [2]. The only difference is that one of the coefficients that appear in the wave system depends on $\mathbf{x}$. Note that (33) is compatible with (31) because $\tilde{\rho}^{(0)}\left(\tilde{a}^{(0)}\right)^2$ is uniform.

Proposition 6. *The LMAAP (for Low Mach Acoustic Accuracy Preserving) scheme [2, 10] associated to a wave system of the form (22), given by (30) with*

$$C_{ij}^{11} = \tilde{a}_{ij}^{(0)}, \quad \mathbf{C}_{ij}^{12} = \varepsilon \frac{\gamma}{2} \tilde{\rho}_{ij}^{(0)} \left(\tilde{a}_{ij}^{(0)}\right)^2 |\mathbf{n}_{ij}|, \quad \mathbf{C}_{ij}^{21} = -\varepsilon \frac{1}{2\gamma \tilde{\rho}_i^{(0)}} |\mathbf{n}_{ij}|^T, \quad C_{ij}^{22} = 0, \tag{34}$$

satisfies the semi-discrete energy inequality (32), where $\varepsilon = \pm 1$ and $|\mathbf{n}_{ij}|$ is a vector which its components are the absolute values of the components of the vector $\mathbf{n}_{ij}$.

Proof. We recall that $\tilde{\rho}_i^{(0)} \left(\tilde{a}_i^{(0)}\right)^2$ does not depend on i. Using Proposition 5, it is sufficient to prove that $\tilde{\rho}_{ij}^{(0)} \left(\tilde{a}_{ij}^{(0)}\right)^2 = \tilde{\rho}^{(0)} \left(\tilde{a}^{(0)}\right)^2$. We first remind that $\rho_{ij}^{(0)} \left(a_{ij}^{(0)}\right)^2 = \rho_{ij}^{(0)} \left(\chi^{(0)} + \kappa^{(0)} H_{ij}^{(0)}\right)$. By developing this equality with the Roe average formula (5) on $H_{ij}^{(0)}$, also using the fact that $\rho_{ij}^{(0)} = \sqrt{\rho_i^{(0)} \rho_j^{(0)}}$ therefore $\rho_{ij}^{(0)} \zeta_{ij}^{(0)} = \rho_i^{(0)} \left(1 - \zeta_{ij}^{(0)}\right)$ and $\rho_{ij}^{(0)} \left(1 - \zeta_{ij}^{(0)}\right) = \rho_j^{(0)} \zeta_{ij}^{(0)}$, one can easily shows that $\rho_{ij}^{(0)} \left(a_{ij}^{(0)}\right)^2$ is a convex combination of $\rho_i^{(0)} \left(a_i^{(0)}\right)^2$ and $\rho_j^{(0)} \left(a_j^{(0)}\right)^2$. That concludes the proof. □

We note that compared to the expression given in [2, 10], the vector $\mathbf{1}_d/\sqrt{d}$ has been replaced by the vector $|\mathbf{n}_{ij}|$ corresponding to taking the absolute value of the normal on each component. This improvement has the advantage of allowing us to find exactly the one-dimensional scheme when studying a unidirectional flow on 2D or 3D Cartesian mesh.

The explicit LMAAP scheme satisfies the von-Neumann necessary condition for stability with the same CFL condition as the Roe scheme [2]. Its artificial diffusion allows us to consider convective but also acoustic low Mach flows, called mixed diffusion in [9]. Only a few mixed diffusion schemes have a non-diagonal diffusion matrix. We refer to [9, 17] for other examples.

3.2 From the wave system to the full Euler system

In order to extend the LMAAP fix from the wave equations to the full Euler equations, we first extend the dimensionless system obtained for the wave equations to dimensionless Euler equations. Then we deduce the expression of the Roe-LMAAP flux.

Definition 1. *(Expression of the Roe-LMAAP scheme associated to the full Euler system) The Roe-LMAAP flux is the following*

$$\Phi_{ij}^{Roe\text{-}LMAAP} = \Phi_{ij}^{Roe} + (1 - \theta_{ij}) \begin{pmatrix} \frac{1}{2a_{ij}}(p_i - p_j) + \varepsilon \frac{\rho_{ij}}{2} (\mathbf{u}_i - \mathbf{u}_j) \cdot |\mathbf{n}_{ij}| \\ -\varepsilon \frac{p_i - p_j}{2} |\mathbf{n}_{ij}| \\ \frac{H_{ij}}{2a_{ij}} (p_i - p_j) + \varepsilon \frac{\rho_{ij} H_{ij}}{2} (\mathbf{u}_i - \mathbf{u}_j) \cdot |\mathbf{n}_{ij}| \end{pmatrix}$$
$$- (1 - \theta_{ij}) \begin{pmatrix} 0 \\ \left(\frac{\rho_{ij} a_{ij}}{2} (\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}\right) \mathbf{n}_{ij} \\ 0 \end{pmatrix}.$$

with $\theta_{ij} = \min(M_{ij}, 1)$, where $M_{ij} = \max(M_i, M_j)$ and $\varepsilon = \pm 1$. Note that it is up to the engineer who will code this numerical flux to choose to assign ε as either 1 or -1, since the numerical results will be similar in both cases.

Note that we have introduced the term $(1 - \theta_{ij})$ to get the original Roe scheme if $\theta_{ij} = 1$. Finally, with this expression, we get that the Roe-LMAAP scheme satisfies both Proposition 3 and Proposition 4. Interestingly, with the new fix, the contact discontinuities are preserved as we add terms involving pressure and velocity jumps.

4 Numerical results

In this section we present some numerical results on the full Euler system (1) equipped with the equation of state

$$p = (\bar{\gamma} - 1)\rho e - (\bar{\gamma} - 1)\rho \bar{q} - \bar{\gamma} \bar{\pi}. \tag{35}$$

dedicated for stiffened gas. Note that for this case $\chi = (1 - \bar{\gamma})\bar{q}$ and $\kappa = \bar{\gamma} - 1$. Furthermore, the value of ε is set to 1 in the expression of the Roe-LMAAP flux, for the subsequent numerical results.

4.1 One dimensional Riemann problem with perfect gas

In this test, we consider the equation of state (35) with the parameters: $\bar{\gamma} = 1.4$, $\bar{q} = 0$, $\bar{\pi} = 0$. The selected test case is a Riemann problem with the following left and right states:

$$\rho_l = 1, \quad \mathbf{u}_l = 0.0001, \quad p_l = 1,$$
$$\rho_r = 1, \quad \mathbf{u}_r = 0.0005, \quad p_r = 1,$$

and final time equals to 0.1 s. Numerical results are presented in Figure 1 for the Roe scheme, the Roe scheme with the Dellacherie et al. fix [5], as well as for the Roe-LMAAP scheme. We observe that oscillatory modes appear on the pressure, density and velocity computed using the Dellacherie et al. fix while there is no oscillation using the new Roe-LMAAP fix.

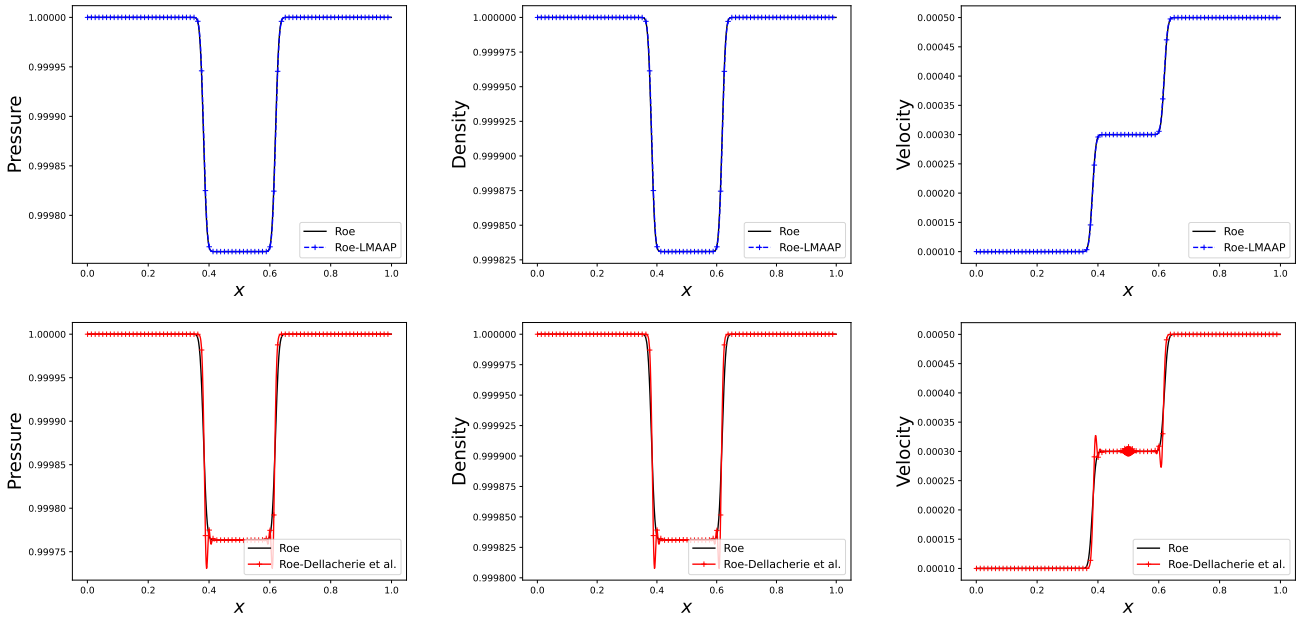


Figure 1: Numerical results for a Riemann problem.

4.2 One dimensional channel with a singular pressure loss

In this test, we consider a fluid governed by the stiffened gas equation of states (35) with the constants $\bar{\gamma} = 2.35$, $\bar{q} = -1167.056 \times 10^3$, $\bar{\pi} = 10^9$ which are the same values as in [1]. These values can be used to represent liquid water in a core of a pressurized water reactor under nominal conditions that is typically $p_0 = 155$ bar and $h_0 = 1300$ kJ/kg. The test consists in a one dimensional channel with the fluid being injected at the bottom $x_{in} = 0$ m with a velocity of $u_{in} = 1$ m/s and enthalpy of $h_{in} = h_0$. The pressure is set at the top $p_{out} = p_0$. Note that we have in this case $\rho_0(p_0, h_0) \approx 716.6$ kg/ m^3. Two source terms are added in the equation of momentum : gravity $g = -9.81$ m/s^2 and a singular pressure loss in $x_0 = 1$ m with coefficient $K_0 = 100$. The momentum equation can thus be written in the following form :

$$\partial_t (\rho u) + \partial_x \left(\rho u^2 + p\right) = \rho g - \frac{1}{2} K_0 \delta_{x_0} \rho u |u|,$$

where $\delta_{x_0}(x)$ denotes the Dirac distribution. The spatial discretization of the Dirac term is given by

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{1}{2} K_0 \delta_{x_0} \rho u |u| \mathrm{d}x = \begin{cases} \frac{1}{2} K_0 \rho_i u_i |u_i| & \text{if } x_0 \in [x_{i-1/2}, x_{i+1/2}], \\ 0 & \text{otherwise.} \end{cases}$$

The number of cells is $N = 200$ and the simulation is stopped when the stationary solution is obtained. Results for the Roe scheme with the Dellacherie et al. fix and the Roe-LMAAP scheme are presented on Figure 2. We observe that adding a stiff source term such as a singular pressure loss introduces severe spatial oscillations in the velocity and pressure profiles in the case of the Dellacherie et al. fix. This is not the case for the new Roe-LMAAP scheme. Moreover, one can see that the pressure level before the pressure loss is better captured with the new scheme : this is due to the fact that the source term is evaluated with value of velocity at the center of the cell. Thus, the presence of oscillations disrupts the evaluation of the pressure loss and therefore the pressure profile.

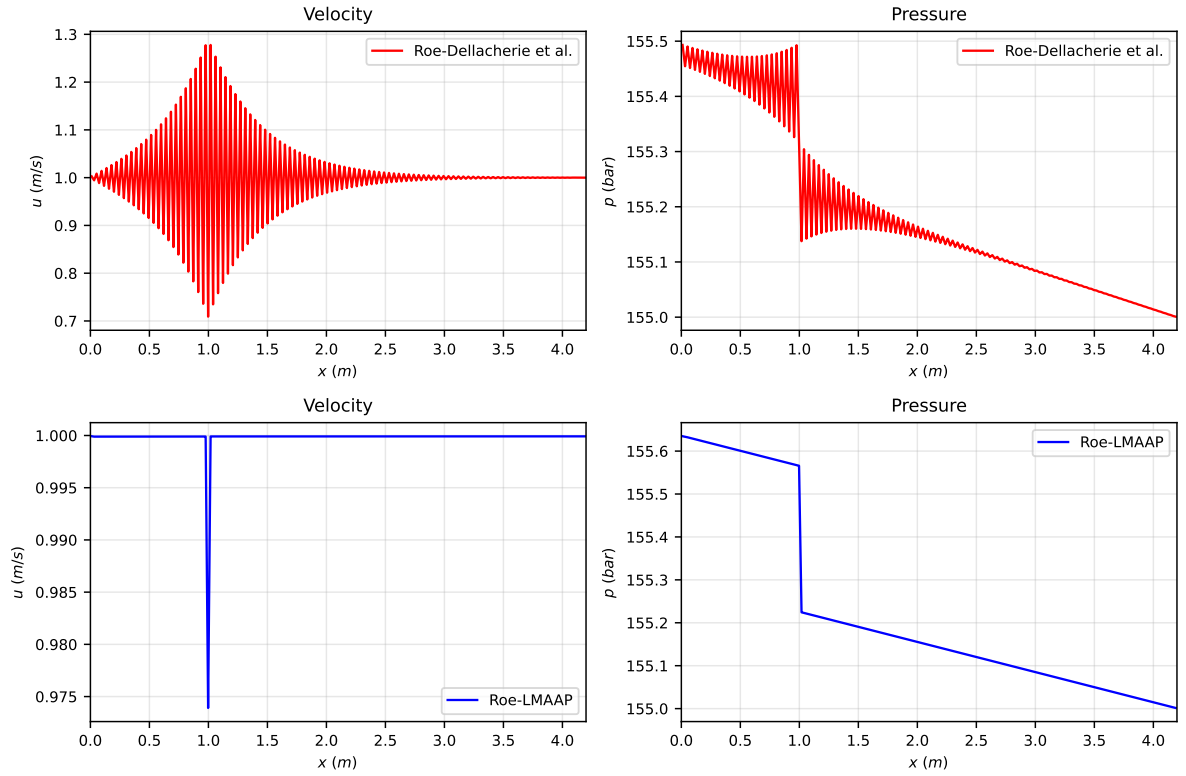


Figure 2: Numerical results for the one dimensional channel with a singular pressure loss.

4.3 Interaction of an acoustic wave with a stationary vortex in a low Mach number flow

In this test, we use the equation of state (35) with the same parameters as for the previous test. The initial condition considered is a superposition of an isentropic vortex with an acoustic wave, given by

$$\mathcal{U}_0 = \begin{cases} \mathcal{U}_0^{\text{Acoustic}} & \text{if } -1 \leq x \leq -0.25, \\ \mathcal{U}_0^{\text{Stationary}} & \text{otherwise.} \end{cases}$$

The domain is supposed to be rectangular $[-1;1] \times [0;1]$, and the initial acoustic wave is computed on the left of the domain while the stationary vortex on the right.

The isentropic vortex is centered in $(0.5, 0.5)$. All the variables are supposed to depend only on the radius $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}/r_0$ where r_0 is a parameter that we define to fix the length of the vortex. For this test case we choose $r_0 = 0.15$. The orthoradial velocity is given by

$$\mathbf{u}_0^{\text{Stationary}} = \left(-u_\infty \left(\frac{y-0.5}{r} \right) \exp\left(\frac{1-r^2}{2} \right), u_\infty \left(\frac{x-0.5}{r} \right) \exp\left(\frac{1-r^2}{2} \right) \right)^T,$$

while the initial pressure and density are respectively given by

$$p_0^{\text{Stationary}} = (p_\infty + \bar{\pi}) \left(1 - \frac{(\bar{\gamma}-1) u_\infty \rho_\infty e^{1-r^2}}{2\bar{\gamma}(p_\infty + \bar{\pi})} \right)^{\bar{\gamma}/(\bar{\gamma}-1)} - \bar{\pi},$$

$$\rho_0^{\text{Stationary}} = \rho_\infty \left(\frac{p + \bar{\pi}}{p_\infty + \bar{\pi}} \right)^{1/(\bar{\gamma}-1)}.$$

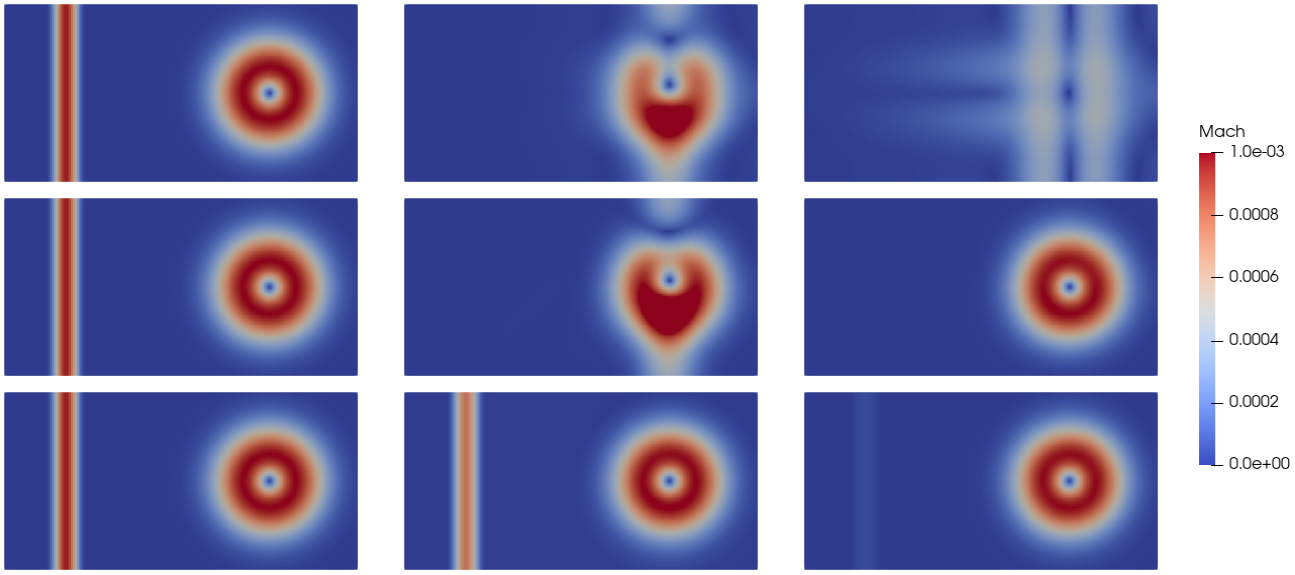
The plane acoustic wave is defined by

$$p_0^{\text{Acoustic}} = \begin{cases} p_\infty \left(1 + M_\infty A e^{1/((x+0.5)(y+0.8))} e^{-1/((-0.65+0.5)(-0.65+0.8))} \right) & -0.8 \leq x \leq -0.5 \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho_0^{\text{Acoustic}} = \rho_\infty \left(\frac{p + \bar{\pi}}{p_\infty + \bar{\pi}} \right)^{1/\bar{\gamma}},$$

$$\mathbf{u}_0^{\text{Acoustic}} = \left(u_\infty + \frac{2c_\infty}{\bar{\gamma}-1} \left(\frac{p + \bar{\pi}}{p_\infty + \bar{\pi}} \right)^{(\bar{\gamma}-1)/2\bar{\gamma}-1}, 0 \right)^T,$$

where $u_\infty = \bar{\gamma} M_\infty (p_\infty + \bar{\pi})/\rho_\infty$ and $c_\infty = \sqrt{\bar{\gamma}(p_\infty + \bar{\pi})/\rho_\infty}$. We take $M_\infty = 10^{-3}$, $p_\infty = 155.10^5$, $\rho_\infty = 716.6$ and the amplitude of the wave $A = 150$. The boundary conditions are supposed to be of type Steger-Warming. Results are shown in Figure 3 for Roe, Roe-LMAAP and Roe-Turkel [8] schemes at different times. The CFL number considered for these tests is 0.3 for Roe and Roe-LMAAP schemes and 0.0003 for Roe-Turkel scheme ; and the mesh type considered is a Cartesian mesh containing 20 000 cells. We observe that when the acoustic wave has left the domain, the vortex is well preserved with the Roe-LMAAP fix and dissipated with the Roe scheme. Whereas, with the Roe-Turkel scheme the acoustic wave is damped without moving and the vortex is preserved.



$t = 0$ s $\qquad$ $t = 0.00063$ s $\qquad$ $t = 0.01$ s

Figure 3: Mach number obtained for the Roe scheme (top), the Roe-LMAAP scheme (middle) and the Roe-Turkel scheme (bottom) for the test case of a stationary vortex combined with a low Mach acoustic wave.

Conclusion

In this paper, we have proposed a low-Mach number fix for Roe's approximate Riemann solver applied for the numerical resolution of the full Euler system. Numerical results show that

- the preconditioning-type fixes do not allow to propagate low Mach acoustic waves,
- the Roe fixes which are consistent with pressure-centered discretization for a wave system introduce oscillatory modes,
- the Roe-LMAAP fix is able to deal with low Mach acoustic computations and reduces oscillatory modes in the numerical solution even when a source term is introduced.

We recall that in the present study, for the asymptotic analysis of the compressible Euler equations, we have considered a general equation of state of the form $p = p(\rho, \rho e)$, however, for the Roe scheme we made some assumptions on the coefficients of the equation of state. The first-order term of the pressure in the Mach number expansion is supposed to be a linear function of the first order terms of ρ and ρe. Further work has to be done for the extension of this study to a more general equation of state. Furthermore, we highlight that via an asymptotic analysis of the full Euler equations, a link with a wave system with non-constant coefficients was derived. In counterpart, performing an asymptotic analysis on the Barotropic Euler equations leads also to a link with a wave system which is however with constant coefficients. More research about the stability of a wave system with non-constant coefficients is required.

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