Existence of financial equilibrium with differential information: the no-arbitrage characterization

Lionel De Boisdeffre

Working Papers Series

Centre d'Analyse Théorique et de Traitement des données économiques

CATT WP No. 5
November 2015

EXISTENCE OF FINANCIAL EQUILIBRIUM WITH DIFFERENTIAL INFORMATION: THE NO-ARBITRAGE CHARACTERIZATION

Lionel de BOISDEFFRE

CATT-UPPA
UFR Droit, Economie et Gestion
Avenue du Doyen Poplawski - BP 1633
64016 PAU Cedex
Tél. (33) 5 59 40 80 01
Internet : **http://catt.univ-pau.fr/live/**

Existence of financial equilibrium with differential information: the no-arbitrage characterization

Lionel de Boisdeffre,[1]

(October 2015)

Abstract

In Cornet-De Boisdeffre (J Math Econ 38: 393-410, 2002), we had extended the classical equilibrium and arbitrage concepts of symmetric information to an asymmetric information model dropping Radner's (Econometrica 47: 655-678, 1979) rational expectations' assumption. In Cornet-De Boisdeffre (Econ Theory 38: 287-293, 2009), we showed how agents could infer enough information, in this model, to preclude arbitrage from financial markets. In De Boisdeffre (Econ Theory 31: 255-269, 2007), we extended to that model Cass' (CARESS WP 84-09, 1984) classical existence theorem for nominal assets, by showing the existence of equilibrium was characterized by a general no-arbitrage condition. We now display the same characteristic property for numeraire assets and, thus, extend Geanakoplos-Polemarchakis' (Essays in Honnor of K.J. Arrow, Starr & Starrett ed., Cambridge UP Vol. 3, 65-96, 1986) classical theorem to the asymmetric information setting. Contrasting with Radner's, these results show that symmetric and asymmetric information economies can be embedded into a common model, where they share similar properties.

Key words: general equilibrium, asymmetric information, arbitrage, existence.

JEL Classification: D52.

[1] CATT-University of Pau, 1 Av. du Doyen Poplawski, 64000 Pau, France.
CES-University of Paris 1, 106-112 Bd. de l'Hôpital, 75013 Paris, France.
Email: lionel.de.boisdeffre@wanadoo.fr

1 Introduction

When agents are asymmetrically informed, they may infer information from observing prices or trade volumes on markets. A traditional approach of that inference problem is given by Radner's (1979) rational expectation model, along which "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents may infer private information of other agents from comparing actual prices and price expectations with theoretical values at a price revealing equilibrium. As is well known, this demanding assumption is only consistent with the generic existence of equilibrium under asymmetric information.

Our approach does not use Radner's assumption. In Cornet-De Boisdeffre (2002), we drop rational expectations and provide the basic tools, concepts and properties for an arbitrage theory, embedding jointly the symmetric and asymmetric information settings, into a same model. In De Boisdeffre (2007), we prove that a financial equilibrium with nominal assets exists in this model, not only generically - as with rational expectations - but under the same no-arbitrage condition, whether agents had symmetric or asymmetric information. This condition characterizes existence of equilibrium, as already known in the symmetric informaton case, since Cass (1984).

This condition may be reached. We show in Cornet-De Boisdeffre (2009) that agents with no price model may always infer enough information, from observing trade on financial markets, to preclude arbitrage. Whence reached, this information could not be refined in our model. Whereas equilibrium always exists, equilibrium prices convey no additional information.

We now show the Cornet-De Boisdeffre (2002) no-arbitrage condition also characterizes the existence of financial equilibrium on numeraire asset markets. This result extends Geanakoplos-Polemarchakis' (1986) theorem of symmetric information.

Formally, the model we present is a two-period pure exchange economy, where agents, possibly asymmetrically informed, face uncertainty, at the first period, on which state of nature will randomly prevail tomorrow, out of a finite state space. They may exchange consumption goods on spot markets, and securities on financial markets, which pay off in numeraire, i.e., in a given commodity (bundle).

The paper is organized as follows: Section 2 introduces the model. Section 3 presents the existence Theorem and its proof. An Appendix proves a Lemma.

2 The basic model

We consider a pure-exchange financial economy with two periods ($t \in \{0,1\}$), finitely many agents, $i \in I := \{1, ..., m\}$, commodities, $l \in \{1, ..., L\}$, states of nature, $s \in S$, and assets, $j \in \{1, ..., J\}$. Agents face uncertainty at the first period ($t = 0$) about which state, $s \in S$, will prevail at the second period ($t = 1$). We shall denote by $s = 0$ the non-random state at $t = 0$ and let $\Sigma' := \{0\} \cup \Sigma$, for any set $\Sigma \subset S$.[2]

[2] Throughout the paper, the scalar product and Euclidean norm are denoted by $\cdot$ and $\|.\|$, respectively. For each $\Sigma \subset S'$, every $S \times J-$matrix $V := (v_j[s])_{(s,j) \in S \times J}$ and $\Sigma \times J-$matrix A, for all collection $(a, a') \in \mathbb{R}^K$, $(p, q, s, l) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J \times \Sigma \times \{1, ..., L\}$ and $(x, x', y, y', z, z') \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^{S'} \times \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma \times (\mathbb{R}^L)^\Sigma \times (\Re^L)^\Sigma$, we denote by:
1) $x[\Sigma]$ and $x'[\Sigma]$, respectively, the truncations of x on $(\mathbb{R}^L)^\Sigma$ and of x' on $\mathbb{R}^\Sigma$;
2) $A[s]$, $y[s]$, $z[s]$, respectively, the row, scalar and vector, indexed by $s \in \Sigma$, of A, y, z;
3) $z^l[s]$ the l^{th} component of $z[s] \in \mathbb{R}^L$ and $z^l := (z^l[s]) \in \mathbb{R}^\Sigma$;
4) $y \leqslant y'$ and $z \leqslant z'$ (resp. $y << y'$ and $z << z'$) the relationships $y[s] \leqslant y'[s]$ and $z^l[s] \leqslant z'^l[s]$ (resp. $y[s] < y'[s]$ and $z^l[s] < z'^l[s]$) for every $s \in \Sigma, l \in \{1, ..., L\}$;
5) $y < y'$ (resp. $z < z'$) the joint relationships $y \leqslant y'$ and $y \neq y'$ (resp. $z \leqslant z'$ and $z \neq z'$);
6) $z_\square z'$ the vector $(z[s] \cdot z'[s]) \in \mathbb{R}^\Sigma$, $y_\square z$ the vector $(y[s]z[s]) \in (\mathbb{R}^L)^\Sigma$;
7) $V(\Sigma)$ and $V(p, \Sigma)$ (when $0 \notin \Sigma$) the $\Sigma \times J-$matrixes defined, respectively, by $V(\Sigma)[s] := V[s]$ and $V(p, \Sigma) := V(p)[s]$, for each $s \in \Sigma$, where $V(p) := ((p[s] \cdot e)v_j[s])$;
8) $W(\Sigma, q)$ and $W(\Sigma, p, q)$ (when $0 \notin \Sigma$) the $\Sigma' \times J-$matrixes defined, respectively, by $W(\Sigma, q)[0] := W(\Sigma, p, q)[0] := -q$, and by $W(\Sigma, q)[s] := V[s]$ and $W(\Sigma, p, q)[s] := V(p)[s]$ for every $s \in \Sigma$. We let $W(q) := W(S, q)$ and $W(p, q) := W(S, p, q)$;
9) $(\mathbb{R}^L_+)^\Sigma := \{x \in (\mathbb{R}^L)^\Sigma : x \geqslant 0\}$, $\mathbb{R}^\Sigma_+ := \{x \in \Re^\Sigma : x \geqslant 0\}$, $(\mathbb{R}^L_{++})^\Sigma := \{x \in (\mathbb{R}^L)^\Sigma : x >> 0\}$, $\mathbb{R}^\Sigma_{++} := \{x \in \mathbb{R}^\Sigma : x >> 0\}$.

2.1 Information structures and refinements

At $t = 0$, each agent, $i \in I$, receives a private signal, or information set $S_i \subset S$, which correctly informs her that an arbitrary state $s \in S_i$ will prevail at $t = 1$. The set collection, (S_i), is the initial information structure. Costlessly, we assume that $\cup_{i=1}^m S_i = S$, and we let $\underline{\mathbf{S}} := \cap_{i=1}^m S_i$ be the pooled information set. Agents may, then, refine their information from observing markets. A collection, (Σ_i), of m subsets of S is called an information structure, or structure, if $\cap_{i=1}^m \Sigma_i \neq \emptyset$, and Γ denotes their set. A structure, $(\Sigma_i) \in \Gamma$, is said to be a (self-attainable) refinement of (S_i), if the relations, $\underline{\mathbf{S}} \subset \widetilde{\Sigma}_i \subset \Sigma_i$, hold, for each $i \in I$.

2.2 The financial market

The financial market permits limited transfers across periods and states, via J numeraire assets $j \in \{1, ..., J\}$, whose contingent payoffs, in each state $s \in S$, are denoted by $v_j[s]e$, where $e \in \mathbb{R}^L_+ \backslash \{0\}$ is the numeraire, that is, a fixed bundle of commodities (and we let $\|e\| = 1$ for simplicity), and $v_j[s]$ is a state-dependent quantity. These quantities, defined for each $(s, j) \in S \times \{1, ..., J\}$, yield a $S \times J-$matrix $V := (v_j[s])$, which is of full column-rank (i.e., $J = rankV$) and known by all agents.

Thus, for every price $p \in (\mathbb{R}^L)^{S'}$, the real numbers $(p[s] \cdot e)v_j[s]$, for each $(s, j) \in S \times \{1, ..., J\}$, define a $S \times J$ price-dependent payoff matrix $V(p) := (\ (p[s] \cdot e)v_j[s]\ )$, in units of account, which is of full column-rank, whenever $p[s] \cdot e > 0$ for each $s \in S$. Given the asset price $q \in \mathbb{R}^J$, a portfolio is a vector $z \in \mathbb{R}^J$, tradable for $q \cdot z$ units of account at $t = 0$, which promises delivery of a flow Vz of contingent payoffs in numeraire at $t = 1$.

We now define and characterize arbitrage.

Definition 1 *Let a structure,* $(\Sigma_i) \in \Gamma$*, be given. A price,* $q \in \mathbb{R}^J$*, is said to be a common no-arbitrage price of the structure* (Σ_i)*, or the structure* (Σ_i) *to be* q*-arbitrage-free, if one of the following equivalent assertions holds:*

(a) $\nexists (i, z) \in I \times \mathbb{R}^J : W(\Sigma_i, q)z > 0$*;*

(b) $\nexists (i, z, p) \in I \times \mathbb{R}^J \times (\mathbb{R}^L)^{S'} : W(\Sigma_i, p, q)z > 0$ *and* $p[s] \cdot e > 0, \forall s \in S$*;*

(c) $\forall i \in I, \exists \lambda_i \in \mathbb{R}^{\Sigma_i}_{++}$ *(called individual state price), such that* $q = {}^t\lambda_i V(\Sigma_i)$*.*

We denote by $Q_c[(\Sigma_i)]$ *the set of common no-arbitrage prices of* (Σ_i)*.*

The structure (Σ_i) *is said to be arbitrage-free if* $Q_c[(\Sigma_i)] \neq \varnothing$*.*

Remark 1 The equivalence between the above Assertions (a) and (b) is immediate. That between (a) and (c) is standard (see, e.g., Magill & Quinzii, 1996).

Claim 1 characterizes arbitrage-free structures, whose proof is given, mutatis mutandis, in Cornet-De Boisdeffre (2002), p. 401, to which we refer the reader.

Claim 1 *A structure,* $(\Sigma_i) \in \Gamma$*, is arbitrage-free if and only if it meets the following "AFAO" Condition:* $\nexists [j, (z_i)] \in I \times (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0$, $V(\Sigma_j)z_j > 0$ *&* $V(\Sigma_i)z_i \geqslant 0, \forall i \in I$*.*

Remark 2 It follows from Cornet-De Boisdeffre (2009), Theorem 3, that agents can always infer an arbitrage-free refinement of (S_i), from observing financial trade.

2.3 The commodity market

Commodities may be traded on spot markets, or consumed, at both dates. The generic agent, $i \in I$, has $X_i := (\mathbb{R}^L_+)^{S'_i}$ for consumption set. She has an endowment, $e_i \in X_i$, and a preference correspondence, P_i, represented by a utility function, $u_i : X_i \to \mathbb{R}$, and defined, for every $x \in X_i$, by the set, $P_i(x) := \{y \in X_i : u_i(y) > u_i(x)\}$, of consumptions, which she strictly prefers to x. It is to bound below the value of the numeraire (see Lemma 1 below), that the generic agent's preferences are ordered and separable. That is, there exist indexes, $v_i^s : \mathbb{R}_+^2 \to \mathbb{R}$ (for each $s \in S_i$), such that $u_i(x) := \sum_{s\in S_i} v_i^s(x[0], x[s])$, for every $x \in X_i$.

Given prices, $(p,q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, and using the notations in footnote, all agents, $i \in I$, have for budget, attainable allocation and attainable strategy sets, respectively:

$B_i(p,q) := \{\ (x,z) \in X_i \times \mathbb{R}^J :\ p[S_i'] \,\Box\, (x\text{-}e_i) \leqslant W(S_i,p,q)z\ \};$

$\mathcal{A} := \{x := (x_i) \in \times_{i=1}^m X_i : \sum_{i=1}^m (x_i - e_i)[\underline{\mathbf{S}}'] = 0\};$

$\mathcal{A}(p,q) := \{[(x_i,z_i)] \in \times_{i=1}^m B_i(p,q) : (x_i) \in \mathcal{A},\ \sum_{i=1}^m z_i = 0\}.$

2.4 Agents' behavior and the concept of equilibrium

The economy described above for a given payoff matrix, V, and structure, (S_i), of information signals, $S_i \subset S$, which each agent, $i \in I$, receives (or infers) privately at $t = 0$, is denoted by $\mathcal{E}$. Each agent seeks a strategy, which maximizes the utility of her consumption in the budget set. This yields the following concept of equilibrium.

Definition 2 *A price system, $(p^*,q^*) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, and collection of strategies, $[(x_i^*,z_i^*)] \in \times_{i=1}^m B_i(p^*,q^*)$, is an equilibrium of the economy $\mathcal{E}$, if the following Conditions hold:*

(a) $\forall i \in I,\ B_i(p^*,q^*) \cap P_i(x_i^*) \times \mathbb{R}^J = \varnothing$;

(b) $\sum_{i=1}^m (x_i^* - e_i)[\underline{\mathbf{S}}'] = 0$;

(c) $\sum_{i=1}^m z_i^* = 0$.

Remark 3 We have assumed throughout that, for any state, $s \in S\backslash\underline{\mathbf{S}}$, any two agents, whose information sets contain s, have the same anticipation, $p[s] \in \mathbb{R}_+^L$, of the spot price in state s. This assumption is made to simplify exposition. All the model's definitions and results would hold if agents had idiosyncratic price anticipations in the unrealizable states (i.e., $s \in S\backslash\underline{\mathbf{S}}$), as in De Boisdeffre (2007).

The economy, $\mathcal{E}$, is called standard if it meets the following Assumptions:

Assumption A1 (*non satiation in the numeraire in any state*):

$\forall i \in I,\ \forall (x,s) \in X_i \times S'_i,\ u_i(x+e_s) > u_i(x)$, where $e_s[s] := e,\ e_s[S' \backslash \{s\}] := 0$;

Assumption A2 (*strong survival*): $\forall i \in I,\ \forall s \in S'_i,\ e_i[s] >> 0$;

Assumption A3 (*continuity*):

$\forall i \in I,\ \forall \varepsilon > 0, \forall x \in X_i, \exists \eta > 0,\ s.t.\ \overline{x} \in X_i, \|\overline{x}\text{-}x\| < \eta \Longrightarrow |u_i(\overline{x})\text{-}u_i(x)| < \varepsilon$;

Assumption A4 (*strict quasi-concavity*):

$\forall i \in I,\ \forall ((x,y),\lambda) \in X_i^2 \times]0,1[,\ u_i(x+\lambda(y-x)) > \min(u_i(x), u_i(y))$.

We henceforth assume the economy, $\mathcal{E}$, is standard.

This paper shows a standard economy, $\mathcal{E}$, admits an equilibrium if, and only if, its information structure is arbitrage-free. Namely, the minimum requirement for existence is also a sufficient condition. This outcome departs from the generic existence of a fully-revealing rational expectations' equilibrium along Radner (1979). First, Claim 2 shows that the information structure needs be arbitrage-free at equilibrium.

Claim 2 *Let prices,* $(p,q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, *and strategies,* $[(x_i, z_i)] \in \times_{i=1}^m B_i(p,q)$, *meet Condition* (a) *of Definition 2 of equilibrium, then,* $q \in Q_c[(S_i)]$.

Proof Let prices, $(p,q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, and strategies, $[(x_i, z_i)] \in \times_{i=1}^m B_i(p,q)$, be given, which meet Condition (a) of Definition 2. We let the reader check, from Assumption *A1*, that the relation $p[s] \cdot e > 0$ holds, for each $s \in S$. Then, the proof is identical to that of Claim 1 in De Boisdeffre (2007), to which we refer the reader. □

Along Remark 2, we henceforth assume, at no cost, that the initial structure, (S_i), is arbitrage-free, and represents agents' final information at the time of trading. We can now state and prove our main Theorem along a fixed point-like argument.

3 The existence Theorem

Theorem 1 *The standard economy, $\mathcal{E}$, whose information structure, (S_i), is arbitrage-free, admits an equilibrium, $((p,q),[(x_i,z_i)]) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J \times (\times_{i=1}^m B_i(p,q))$. Moreover, the relation $\overline{p}[s]\cdot e > 0$ holds, for each $s \in S'$.*

3.1 Bounding strategies

Lemma 1, below, permits to bound strategies. Before stating the Lemma, we introduce the following sets; first, for each $i \in I$:

$Z_i^o := \{z \in \mathbb{R}^J : V[s]\cdot z = 0, \forall s \in S_i\}$ and $Z_i^{o\perp}$, its orthogonal;

$Z^o := \sum_{i=1}^m Z_i^o$ and $Z^{o\perp} = \cap_{i\in I} Z_i^{o\perp}$, its orthogonal.

Then, we set $\varepsilon \in]0, \frac{1}{L}[$ as given and consider the following nonempty price sets:

$\Delta := \{p \in (\mathbb{R}^L)^{S'} \colon \ \|p[s]\| \leqslant 1, \forall s \in S', p^l[\overline{s}] \geqslant \varepsilon, \forall (l,\overline{s}) \in \{1,...,L\} \times S \backslash \underline{\mathbf{S}}\}$;

$\Delta_\delta := \{p \in \Delta : p[s]\cdot e \geqslant \delta, \forall s \in \underline{\mathbf{S}}\}$, for each $\delta \in]0, \varepsilon[$;

$Q := \{q \in Z^{o\perp} : \|q\| \leqslant 1\}$, $\Pi := \Delta \times Q$ and $\Pi_\delta := \Delta_\delta \times Q$.

Then, denoting by $\mathbf{1}$ the vector of $\mathbb{R}^{S'}$ whose components are all equal to one, we consider, for each $i \in I$ and every $(p,q) \in \Pi$, the following strategy sets:

$\overline{B}_i(p,q) := \{(x,z) \in X_i \times Z_i^{o\perp} : \ p[S_i'] \,\square\, (x - e_i) \leqslant W(S_i,p,q)z + \mathbf{1}\,[S_i']\}$;

$\overline{\mathcal{A}}(p,q) := \{[(x_i,z_i)] \in \times_{i=1}^m \overline{B}_i(p,q) : (x_i) \in \mathcal{A}, \ \sum_{i=1}^m z_i \in Z^o\}$.

Finally, we let, for each $s \in \underline{\mathbf{S}}$:

$$P_s := \{\, p_s \in \mathbb{R}^L, \|p_s\| = 1 : (\exists i \in I, \ \exists (x_i) \in \mathcal{A}, \ s.t. \left[\begin{array}{c} (\ y_i \in P_i(x_i) \ \ and \ \ y_i[S_i' \backslash \{s\}] = x_i[S_i' \backslash \{s\}]\ ) \\ \Rightarrow \ (\ p_s \cdot y_i[s] \geqslant p_s \cdot x_i[s] \geqslant p_s \cdot e_i[s]\ ) \end{array} \right] \}$$

$P := \{p \in \Delta : p[s] \in P_s, \forall s \in \underline{\mathbf{S}}\}$.

Lemma 1 *For the above sets, the following Assertions hold:*

(i) for each $s \in \underline{\mathbf{S}}$, P_s *is closed, hence,* P_s *and* P *are compact sets;*

(ii) there exists $\underline{\delta} \in]0, \varepsilon[$, *such that* $P \subset \Delta_{\underline{\delta}}$*;*

(iii) $\exists \underline{r} > 0 : [(p,q) \in \Pi_{\underline{\delta}}$ *and* $[(x_i, z_i)] \in \overline{\mathcal{A}}(p,q)] \Longrightarrow [\sum_{i=1}^{m}(\|x_i\| + \|z_i\|) < \underline{r}]$*;*

(iv) $\exists r > \underline{r} : [(p,q) \in \Pi,\ [(x_i, z_i)] \in \overline{\mathcal{A}}(p,q)$ *and* $\|z_i\| \leqslant \underline{r},\ \forall i \in I] \Longrightarrow [\sum_{i=1}^{m}(\|x_i\| + \|z_i\|) < r]$.

Proof: See the Appendix. □

We henceforth set $r > 0$ and $\underline{r} > 0$ as given, which meet the Conditions of the above Lemma and let, for every $(i, (p,q)) \in I \times \Pi$:

$X_i^* := \{x \in X_i : \|x\| \leqslant r\}$ and $Z_i^* := \{z \in Z_i^{o\perp} : \|z\| \leqslant \underline{r}\}$;

$B_i^*(p,q) := B_i(p,q) \cap (X_i^* \times Z_i^*)$;

$\mathcal{A}^*(p,q) := \{[(x_i, z_i)] \in \times_{i=1}^{m} B_i^*(p,q)\colon\ (x_i) \in \mathcal{A},\ (\sum_{i=1}^{m} z_i) \in Z^o\}$.

3.2 The existence proof

Following Florenzano (1999), we now prove the existence of equilibrium, along De Boisdeffre (2007). Thus, for each $i \in \{1, ..., m\}$ and each $(p,q) \in \Pi$, we let:

$B_i'(p,q) := \{(x,z) \in X_i^* \times Z_i^* : p[S_i'] \,\square\, (x\text{-}e_i) \leqslant W(S_i, p, q)z + \gamma_{(p,q)}[S_i']\}$;

$B_i''(p,q) := \{(x,z) \in X_i^* \times Z_i^* : p[S_i'] \,\square\, (x\text{-}e_i) << W(S_i, p, q)z + \gamma_{(p,q)}[S_i']\}$,

where $\gamma_{(p,q)} \in \mathbb{R}_+^{S'}$ is defined by: $\gamma_{(p,q)}[0] := 1 - \min(1, \|p[0]\| + \|q\|)$,

$\gamma_{(p,q)}[s] := 1 - \|p[s]\|$ for every $s \in \underline{\mathbf{S}}$ and $\gamma_{(p,q)}[S \backslash \underline{\mathbf{S}}] := 0$.

We introduce a fictitious agent, $i = 0$, representing the market and a reaction correspondence, Ψ_i, for each agent, $i \in I \cup \{0\}$, defined on the convex compact set, $\Theta := \Pi \times (\times_{i=1}^{m} X_i^* \times Z_i^*)$, namely, for every $((p,q),(x,z)) \in \Theta$, we let:

$$\Psi_i((p,q),(x,z)) := \left\{ \begin{array}{ll} B'_i(p,q) & \text{if } (x_i,z_i) \notin B'_i(p,q) \\ B''_i(p,q) \cap P_i(x_i) \times Z^*_i & \text{if } (x_i,z_i) \in B'_i(p,q) \end{array} \right\}, \text{ for each } i \in I,$$

and

$$\Psi_0((p,q),(x,z)) := \{(p',q') \in \Pi : (p'\text{-}p) \cdot \textstyle\sum_{i=1}^m (x_i\text{-}e_i) + (q'\text{-}q) \cdot \sum_{i=1}^m z_i > 0\}.$$

Then, we state successive Properties (Claims 3 to 12, below), whose proofs are all given, mutatis mutandis, in De Boisdeffre (2007), to which we refer the reader.

Claim 3 *For every* $i \in I$, *and every* $(p,q) \in \Pi$, $B''_i(p,q) \neq \varnothing$.

Claim 4 *For every* $i \in I$, B''_i *is convex-valued and lower semicontinuous.*

Claim 5 *For every* $i \in I$, B'_i *is convex-valued and upper semicontinuous.*

Claim 6 *For every* $i \in \{0\} \cup I$, Ψ_i *is lower semicontinuous.*

Claim 7 *There exists* $((p^*,q^*),(x^*,z^*)) \in \Theta$, *such that:*

(i) $\forall (p,q) \in \Pi,\ (p^* - p) \cdot \sum_{i=1}^m (x^*_i - e_i) + (q^* - q) \cdot \sum_{i=1}^m z^*_i \geqslant 0$;

(ii) $\forall i \in I,\ (x^*_i, z^*_i) \in B'_i(p^*,q^*)$ *and* $B''_i(p^*,q^*) \cap P_i(x^*_i) \times Z^*_i = \varnothing$.

Claim 8 $\overline{z}^* := (\sum_{i=1}^m z^*_i) \in Z^o$.

Remark 4 Since the relation $Q \subset Z^{o\perp}$ holds, from Claim 8, Claim 7-(i) may now be written $(p^* - p) \cdot \sum_{i=1}^m (x^*_i\text{-}e_i) \geqslant 0$, for all $p \in \Delta$, and there exists $z' := (z'_i) \in \times_{i=1}^m Z^o_i$ such that $\sum_{i=1}^m z^*_i = \sum_{i=1}^m z'_i$. For each $i \in I$, we henceforth let $z_i := z^*_i - z'_i$, which satisfy $W(\underline{\mathbf{S}}', p^*, q^*) z^*_i = W(\underline{\mathbf{S}}', p^*, q^*) z_i$ (since $q^* \in Z^{o\perp}_i$ and $z'_i \in Z^o_i$) and $\sum_{i=1}^m z_i = 0$.

Claim 9 $x^* = (x^*_i) \in \mathcal{A}$, *i.e.*, $\sum_{i=1}^m (x^*_i - e_i)[\underline{\mathbf{S}}'] = 0$.

Claim 10 $[(x^*_i, z^*_i)] \in \overline{\mathcal{A}}(p^*,q^*)$, *hence*, $\sum_{i=1}^m (\|x^*_i\| + \|z^*_i\|) < r$.

Claim 11 *For each* $i \in I$, (x^*_i, z^*_i) *is optimal in* $B'_i(p^*,q^*)$.

Claim 12 $\gamma_{(p^*,q^*)} = 0$, *that is*, $B'_i(p^*,q^*) = B^*_i(p^*,q^*)$, *for each* $i \in I$.

The following Claims are proved directly, differing from De Boisdeffre (2007).

Claim 13 $\forall s \in S \backslash \underline{\mathbf{S}}$, $\|p^*[s]\| \geqslant \varepsilon$, $\forall s \in \underline{\mathbf{S}}, \|p^*[s]\| = 1$, $\forall s \in S'$, $p^*[s] \cdot e > 0$.

Proof Let $s \in S'$ be given. From Claim 10 and Assumptions *A1-A4*, there exist $i \in I$, such that $s \in S'_i$ (say $i = 1$), and also $x_1 \in P_1(x^*)$ and $\lambda > 0$, such that $\|x_1\| < r$, $x_1[S'_1 \backslash \{s\}] = x^*_1[S'_1 \backslash \{s\}]$ and $x_1[s] = x^*_1[s] + \lambda e$. Hence, $p^*[s] \cdot e > 0$; otherwise, Claim 7-(ii) would imply $(x_1, z^*_1) \in B'_1(p^*, q^*)$ and contradict Claim 11. If $s \notin \underline{\mathbf{S}}'$, the definition of Π and $(p^*, q^*) \in \Pi$ yield $\|p^*[s]\| \geqslant \varepsilon > 0$. If $s \in \underline{\mathbf{S}}$, Claim 12 yields $\|p^*[s]\| = 1$. □

Claim 14 $p^* \in \Delta_{\underline{\delta}}$ *and* $((p^*, q^*), [(x^*_i, z_i)])$*, along Remark 4, is an equilibrium of* $\mathcal{E}$*.*

Proof We show, first, that $p^* \in P$, i.e., $p^*[s] \in P_s$, for every $s \in \underline{\mathbf{S}}$. Indeed, from Claim 13, $p^* \in \Delta^* := \{p \in \Delta : \|p^*[s]\| = 1, \forall s \in \underline{\mathbf{S}}\}$ and, from Claim 9, $x^* \in \mathcal{A}$. Let $s \in \underline{\mathbf{S}}$ be given. Referring the reader to Remark 4 and to the proof of Claim 12 (in De Boisdeffre (2007)), one has $V(p^*)[s] \cdot z_i = p^*[s] \cdot (x^*_i - e_i)[s]$, for each $i \in I$, with $\sum_{i=1}^m z_i = 0$. Thus, there exists $i \in I$, such that $V(p^*)[s] \cdot z_i \geqslant 0$, and we let the reader check from Claims 10, 11 and 12, and from Assumption *A4*, that the triple $(p[s], (x^*_i), i)$ meets the conditions of the definition of P_s. Hence, $p^* \in P$, which implies, from Lemma 1 and Claim 10, that $p^* \in \Delta_{\underline{\delta}}$, and $\sum_{i=1}^m (\|x^*_i\| + \|z^*_i\|) < \underline{r} < r$.

From Claims 10 and 12 and Remark 4, the collection $((p^*, q^*), [(x^*_i, z_i)])$ belongs to $\Pi \times (\times_{i=1}^m B_i(p^*, q^*))$ and meets Conditions (b) and (c) of Definition 2 of equilibrium. Assume, by contraposition, that it does not meet Condition (a). Then, there exist $i \in I$ and $(x_i, \widetilde{z}_i) \in B_i(p^*, q^*) \cap P_i(x^*_i) \times Z_i^{0\perp}$. From Assumption *A4*, the above relations, $\|z^*_i\| + \|x^*_i\| < \underline{r} < r$, and the convexity of $B^*_i(p^*, q^*)$ and $P_i(x^*_i) \times Z_i^{0\perp}$, we may take $(x_i, \widetilde{z}_i)$ close enough to (x^*_i, z^*_i) so that $(x_i, \widetilde{z}_i) \in B^*_i(p^*, q^*) \cap P_i(x^*_i) \times Z^*_i$, which contradicts Claims 11-12. This contradiction proves $((p^*, q^*), [(x^*_i, z_i)])$ is an equilibrium of $\mathcal{E}$. □

Appendix

Recalling notations in footnote and Section 3, we let for each $s \in \underline{\mathbf{S}}$:

$$P_s := \{\, p_s \in \mathbb{R}^L, \|p_s\| = 1 : (\exists i \in I,\ \exists (x_i) \in \mathcal{A},\ \ s.t. \left[\begin{array}{c} (\ y_i \in P_i(x_i)\ \ and\ \ y_i[S'_i \backslash \{s\}] = x_i[S'_i \backslash \{s\}]\ ) \\ \Rightarrow\ (\ p_s \cdot y_i[s] \geqslant p_s \cdot x_i[s] \geqslant p_s \cdot e_i[s]\ ) \end{array} \right] \}$$

$$P := \{p \in \Delta : p[s] \in P_s, \forall s \in \underline{\mathbf{S}}\}.$$

Lemma 1 *For the above sets, the following Assertions hold:*

(i) for each $s \in \underline{\mathbf{S}}$, P_s is closed, hence, P_s and P are compact sets;

(ii) there exists $\underline{\delta} \in]0, \varepsilon[$, such that $P \subset \Delta_{\underline{\delta}}$;

(iii) $\exists \underline{r} > 0 : [(p,q) \in \Pi_{\underline{\delta}}$ and $[(x_i, z_i)] \in \overline{\mathcal{A}}(p,q)] \Longrightarrow [\sum_{i=1}^m (\|x_i\| + \|z_i\|) < \underline{r}]$;

(iv) $\exists r > \underline{r} : [(p,q) \in \Pi,\ [(x_i, z_i)] \in \overline{\mathcal{A}}(p,q)$ and $\|z_i\| \leqslant \underline{r},\ \forall i \in I] \Longrightarrow [\sum_{i=1}^m (\|x_i\| + \|z_i\|) < r]$.

Proof (i) Let $s \in \underline{\mathbf{S}}$ and a converging sequence $\{p^k\}_{k \geqslant 1} := \{(p^k_{\overline{s}})_{\overline{s} \in S'}\}_{k \geqslant 1}$ of elements of P be given. Since Δ is closed, there exists $p \in \Delta$, s.t. $p_s := \lim p^k_s = p[s]$. Moreover, w.l.o.g., we may assume there exist $i \in I$ and a sequence $\{x^k\}_{k \geqslant 1} := \{(x^k_i)\}_{k \geqslant 1}$ of elements of $\mathcal{A}$, converging to some $x := (x_i)$ in $cl\mathcal{A}$, the closure of $\mathcal{A}$ in $(\mathbb{R}_+ \cup \{+\infty\})^{LS'm}$, such that, for each $k \geqslant 1$, (p^k_s, i, x^k) satisfies the conditions of the definition of P_s. We let the reader check, as standard, from market clearance conditions, that the sequence, $\{x^k[\underline{\mathbf{S}}']\}_{k \geqslant 1} := \{(x^k_i[\underline{\mathbf{S}}'])\}_{k \geqslant 1}$, is bounded, hence, $x[\underline{\mathbf{S}}'] := (x_i[\underline{\mathbf{S}}'])$ is finite.

For every $k \geqslant 1$, let $\widetilde{x}^k := (\widetilde{x}^k_i) \in \mathcal{A}$ be defined by $\widetilde{x}^k[s] := x[s]$, $\widetilde{x}^k[0] := x[0]$ and $\widetilde{x}^k_i[s_i] := x^k[s_i]$, for every pair $(i, s_i) \in I \times S_i \backslash \{s\}$. Then, the relations $p^k_s \cdot (x^k_i[s] - e_i[s]) \geqslant 0$, for every $k \geqslant 1$, yield, in the limit, $p_s \cdot (\widetilde{x}^k_i[s] - e_i[s]) := p_s \cdot (x_i[s] - e_i[s]) \geqslant 0$.

We now show there exists $k \geqslant 1$, such that $(p_s, i, \widetilde{x}^k)$ satisfies the conditions of the definition of P_s (hence, $p_s := \lim p^k_s \in P_s$, i.e., P_s is closed).

Assume, by contraposition, that this is not the case. Then, from above, for every $k \geqslant 1$, there exists $y_i^k \in P_i(\widetilde{x}^k)$, s.t. $y_i^k[S_i' \backslash \{s\}] = \widetilde{x}_i^k[S_i' \backslash \{s\}]$ and $p_s \cdot (y_i^k[s] - x_i[s]) < 0$. Hence, $v_i^s(x_i[0], y_i^k[s]) > v_i^s(x_i[0], x_i[s])$. Let $k \geqslant 1$ be given. Then, the latter inequalities and Assumption *A3* insure the existence of $K \geqslant k$, such that, for every $k' \geqslant K$, $y_i^k \in P_i(x^{k'})$, which implies, by construction of each $x^{k'}$, $p_s^{k'} \cdot (y_i^k[s]\text{-}x_i^{k'}[s]) \geqslant 0$, hence, in the limit $(k' \to \infty)$, $p_s \cdot (y_i^k[s]\text{-}x_i[s]) \geqslant 0$. This contradicts the above inequality $p_s \cdot (y_i^k[s]\text{-}x_i[s]) < 0$. Hence, $p_s \in P_s$, and P_s and P are closed, proving Lemma 1-(i).[3] $\square$

(ii) Let $s \in \underline{\mathbf{S}}$ be given. We prove, first, that $p[s] \cdot e > 0$ for every $p \in P$. Indeed, let $p \in P$ and $(p[s], i, x) \in P_s \times I \times \mathcal{A}$ meet the conditions of the definition of P_s. From Assumption *A2*, there exists $a_i \in X_i$ such that, $a_i[S_i' \backslash \{s\}] := 0$, and $p[s] \cdot a_i[s] < p[s] \cdot e_i[s] \leqslant p[s] \cdot x_i[s]$. Then, for every $n > 1$, we let $x_i^n := (\frac{1}{n} a_i + (1 - \frac{1}{n}) x_i) \in X_i$, which satisfies $p[s] \cdot x_i^n[s] < p[s] \cdot x_i[s]$ by construction. Referring to Assumptions *A1*-*A3* and their notations, there exists $n > 1$, such that $y := (x_i^n + (1 - \frac{1}{n}) e_s) \in P_i(x_i)$, which implies, $p[s] \cdot x_i[s] \leqslant p[s] \cdot y[s] = p[s] \cdot (x_i^n[s] + (1 - \frac{1}{n}) e) < p[s] \cdot x_i[s] + (1 - \frac{1}{n}) p[s] \cdot e$. Hence, $p[s] \cdot e > 0$ and, for every pair $(p, s) \in P \times \underline{\mathbf{S}}$, there exists $\delta_{p_s}^s \in]0, \varepsilon[$, such that $p[s] \cdot e > \delta_p^s$. The mapping $\varphi_s : P \to \mathbb{R}_{++}$, defined by $\varphi_s(p) := p[s] \cdot e$ is continuous and attain its minimum for some element $\underline{p_s}$ of the compact set P. The reader will readily check that $\underline{\delta} := \min \delta_{\underline{p_s}}^s$, for $s \in \underline{\mathbf{S}}$, satisfies $P \subset \Delta_{\underline{\delta}}$. This proves Lemma 1-$(ii)$. $\square$

(iii)-(iv) The proofs of Assertions (iii) and (iv) are similar to that of Lemma 1, p. 266, of De Boisdeffre (2007) and left to the reader. $\square$

[3] The Assumption of separable utilities was only used to prove Lemma 2. With symmetric information, the reader will readily check it is not required (since attainable allocations are bounded) and, moreover, that all paper's proofs remain valid if we use preferences correspondences (instead of utility functions), which are open and convex-valued and replace Assumption *A4* by: $\forall i \in I, \forall (\lambda, (x, y)) \in]0, 1] \times {X_i}^2$, $y \in P_i(x) \Longrightarrow (\lambda y + (1 - \lambda) x) \in P_i(x)$

References

[1] Cass, D. Competitive equilibrium with incomplete financial markets. University of Pennsylvania. CARESS Working Paper No. 84-09 (1984).

[2] Cornet, B., De Boisdeffre, L. Arbitrage and price revelation with asymmetric information and incomplete markets. J.Math Econ 38, 393-410 (2002).

[3] Cornet, B., De Boisdeffre, L. Elimination of arbitrage states in asymmetric information models. Econ Theory 38, 287-293 (2009).

[4] Florenzano, M. General equilibrium of financial markets: An introduction. Université Paris 1, Cahiers de la MSE, Série Bleue No. 1999.66 (1999).

[5] Gale, D., Mas-Colell, A. An equilibrium existence theorem for a general model without ordered preferences. J Math Econ 2, 9-15 (1975).

[6] Gale, D., Mas-Colell, A. Corrections to an equilibrium existence theorem for a general model without ordered preferences. J Math Econ 6, 297-298 (1979).

[7] Geanakoplos, J., Polemarchakis, H. Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete. W. Haller et alii eds., Essays in honnor of K. Arrow, Vol. 3, 65-95, Cambridge U.P. (1986).

[8] Magill, M., Quinzii, M. Theory of incomplete markets. MIT, Cam. UP (1996).

[9] Radner, R. Rational expectations equilibrium: Generic existence and the information revealed by prices. Econometrica 47, 655-678 (1979).