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Abstract

We consider a pure exchange economy, where consumers may exchange commodities, on spot markets, and securities, on purely financial markets, and be asymmetrically informed. Agents have private characteristics, anticipations and beliefs, and no model to forecast prices. Therefore, they face an incompressible uncertainty, represented by a "minimum uncertainty set", which typically adds to the 'exogenous uncertainty', on tomorrow's state of nature, an 'endogenous uncertainty' on spot prices, which depend on agents' private beliefs. At equilibrium, all consumers expect the 'true' price in each realizable state as a possible outcome, and elect optimal strategies, ex ante, which clear on all markets, ex post. We show that equilibrium exists under standard conditions, as long as agents' prior anticipations, which may be refined from observing markets, embed the minimum uncertainty set.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

When agents' information is incomplete or asymmetric, the issue of how they may infer information from observing prices or trade volumes is essential and, yet, debated. Quoting Ross Starr (1989), "*the theory with asymmetric information is not well understood at all. In short, the exact mechanism by which prices incorporate information is still a mystery and an attendant theory of volume is simply missing.*" A traditional response is given by the REE (rational expectations equilibrium) models of asymmetric information, by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the relationships between the information signals in the economy and the equilibrium prices, along a so-called "*forecast function*". Generically in the REE model, equilibrium prices exist and are separating (i.e., differ across the sets of private information signals), hence, fully revealing. That is, agents infer the private information of all other agents from observing such prices. As acknowledged by Radner himself, this assumption presumes much of agents' computational and inference abilities and may prevent equilibrium to exist.

Cornet-De Boisdeffre (2002) suggests an alternative approach to deal with asymmetric information. Our model has two periods with an uncertainty, at the first period, about which state of the world will prevail tomorrow, out of a finite state space. Asymmetric information amongst finitely many agents is represented by a private information signal, which correctly informs each agent, at the first period, that tomorrow's true state will be in a subset of the state space. Agents may exchange finitely many consumption goods on spot markets at both periods. They have preferences over their consumption sets, and receive a conditional endowment in each state they expect. They may also exchange, unrestrictively, finitely many nominal assets, which enable limited financial transfers across periods and states.

The latter model drops Radner's forecast function and extends the classical definitions of equilibrium, prices and arbitrage, into unique broad concepts, which apply to both the symmetric and asymmetric information settings. In particular, the concept of equilibrium is defined as the classical one of symmetric information, but the fact that the set of expected states may differ across agents, and spot markets need only clear in the commonly expected states. In this economy, De Boisdeffre (2007) shows the existence of equilibrium is characterized by the extended no-arbitrage condition of the model. This existence result generalizes Cass' (1984) standard one to the asymmetric information setting. It is stronger than the REE's (which is only generic), and little demanding from agents. As shown by Cornet-De Boisdeffre (2009), this no-arbitrage condition may, indeed, be reached by agents, with no price model, from simply observing exchange opportunities on financial markets.

On actual markets, agents are unlikely to infer information and derive strategies from a forecast function a la Radner (1979). Instead, they would observe, respond and learn from arbitrage opportunities. Competing arbitrageurs would take advantage of consumers' incomplete asymmetric information to sell them, whenever possible, zero-sum bundles of financial portfolios, which some buyers would mistakenly perceive as profitable. This search for profit by arbitrageurs does not require any fine information, but only to act as intermediaries and observe arbitrage. When the prices of such portfolios fall to zero, as a result of (a Bertrand) competition, realistic buyers would infer that some events (initially thought to be possible) cannot occur. Namely, those events through which the desired portfolios would grant an arbitrage. Consumers need no price model, skill or knowledge, to make such inferences.

Cornet-De Boisdeffre (2009) presents their formal definitions and outcomes.

The above papers show that dropping rational expectations, along Radner (1979), enables to picture the transmission of information via markets, to restore a full existence property of equilibrium with asymmetric information, but not to explain how agents forecast prices perfectly in all realizable states. Indeed, these papers still retain Radner's (1972) standard assumption that unobserved prices are uniquely, commonly and perfectly anticipated by agents in such states.

Even with symmetric information, the above "*perfect foresight*" equilibrium, quoting Radner (1982), "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*". That equilibrium of plans, prices and price expectations would be justified if agents had, not only a complete knowledge of all other agents' private characteristics (the primitives of the economy), but the ability to compute equilibrium prices accordingly, a common agreement to elect one particular price (if multiple), despite contradictory interests, and the common knowledge of game theory that no one would deviate from the elected price. The latter requirements are also referred to as rational expectations (RE). Though extreme, RE are standard in the classical theory. The current paper shows that dropping them, in the double sense of Radner (1972, 1979), is not only possible, but may also reconcile the sequential and temporary equilibrium notions into a unique concept, whose full existence is guaranteed under fairly natural conditions.

In the current paper, agents have no forecast function, along Radner, and may keep their own characteristics private: anticipations, information, beliefs, preferences and endowments. We show this privacy typically results in an incompressible uncertainty on future prices, represented by a so-called "*minimum uncertainty set*". We argue the latter set (or a bigger one) might be inferred from observing past prices. The model's sequential equilibrium notion, or "*correct foresight equilibrium*", is defined as Cornet-De Boisdeffre's (2002), but the fact that prices are now anticipated by agents as elements of (possibly uncountable) anticipation sets. In particular, equilibrium prices are no longer uniquely and endogenously determined, but commonly anticipated by agents as elements of their private anticipation sets. From Cornet-De Boisdeffre (2009), we assume, non restrictively, that agents' information is arbitrage-free. Then, we show the following existence Theorem in the case of purely financial markets (real assets are studied in a companion paper): provided every agent's anticipation set included the minimum uncertainty set, equilibrium exists, whatever the probability distributions over anticipation sets.

The above approach to information transmission and the resulting equilibrium seems to picture agents' actual behaviors on markets. Endowed with no price model and unaware of the primitives of the economy, they infer an arbitrage-free refinement of their information from observing trade, first. They have no means of going beyond that refinement. Then, market forces, driven by prices, lead them to equilibrium. That path discards rational expectations, in the two Radner 1972 and 1979 senses.

The paper is organized as follows: Section 2 presents the model. Section 3 states the existence Theorem and discusses its main Assumption. Section 4, and a technical Annex, prove the Theorem and a Lemma.

2 The basic model

We consider, throughout, a two-period economy, with private information signals, a consumption good market and a financial market. The sets, I, S, H and J, respectively, of consumers, states of nature, goods and assets are all finite. The first period will also be referred to as $t = 0$ and the second, as $t = 1$. At $t = 0$, there is an uncertainty on which state of nature, $s \in S$, will prevail tomorrow. The non random state at $t = 0$ is denoted by $s = 0$ and, whenever $\Sigma \subset S$, we also denote $\Sigma' := \{0\} \cup \Sigma$.

This Section is organized as follows: sub-section 2.1 presents markets, information and individual beliefs, and sub-section 2.2 presents agents' behaviours and the concept of equilibrium.

2.1 Markets, information and beliefs

Agents consume and may exchange the same consumption goods, $h \in H$, on the spot markets of each period. The generic i^{th} agent's welfare is measured, ex post, by a utility index, $u_i : \mathbb{R}_+^{2H} \to \mathbb{R}_+$, over her consumptions at both dates.

At the first period, each agent, $i \in I$, has some private information signal, $S_i \subset S$, about which states of the world may occur at the next period. That is, she knows that no state, $s \in S \backslash S_i$, will prevail tomorrow. Each set S_i is assumed to contain the true state (that is, any state that can prevail at $t = 1$). Hence, the pooled information set, denoted by $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, is non-empty. Such a collection of subsets of S, whose intersection is non empty, is called an information structure, or structure. Each agent, $i \in I$, will possibly refine her information set beyond S_i at $t = 0$ from observing markets. A structure, (Σ_i), such that $\Sigma_i \subset S_i$, for each $i \in I$, is called a refinement of (S_i), which we denote $(\Sigma_i) \leq (S_i)$. It is called self-attainable if $\underline{\mathbf{S}} = \cap_{i \in I} \Sigma_i$.

At the first period also, in each expected state, $s \in S_i$, the generic i^{th} agent has a set of anticipations, P_s^i, of the spot prices that may prevail if state s obtains. This set, P_s^i , is assumed to be idiosyncratic, exogenous, private and closed in $P := \{p \in \mathbb{R}_{++}^H : \|p\| = 1\}$[2]. Thus, the agent is only concerned about commodities'

[2] As is standard, $\mathbb{R}_+$ denotes the set of non-negative real numbers and $\mathbb{R}_{++}$ denotes that of strictly positive.

relative prices. She is assumed to be unaware of other agents' characteristics, plans or forecasts, and of the way market prices are determined. Throughout the paper, $\Omega_i := \cup_{s \in S_i} \{s\} \times P_s^i$ is set as given, for each $i \in I$, and referred to as the i^{th} agent's anticipation set, and, moreover, we assume that $\cap_{i \in I} \Omega_i$ is non-empty.

The following definitions are used throuhout. We refer to $\Omega := S \times P$ as the forecast set and denote by ω its generic element. A closed subset of Ω is called an anticipation set. A probability distibution over $(\Omega, \mathcal{B}(\Omega))$, whose support is an anticipation set, is called a belief. A collection of anticipation sets, whose intersection is non-empty, is called an (anticipation) structure, and $\mathcal{AS}$ denotes their set. A collection of beliefs, (π_i), whose supports define an anticipation structure, say $(Q_i) \in \mathcal{AS}$, is called a structure of beliefs. Then, (π_i) is said to support (Q_i), which we denote $(\pi_i) \in \Pi[(Q_i)]$.

Agents may operate financial transfers across states in S' (i.e., across the two periods and across the states of the second period) by exchanging, at $t = 0$, finitely many nominal assets $j \in J$, which pay off, at $t = 1$, conditionally on the realization of the state of nature. These payoffs define a $S \times J$ matrix, V, whose row vector is denoted by $V(s) \in \mathbb{R}^J$ in every state $s \in S$. Anticipating the model's subsequent extension to a financial economy with both nominal and real assets, V will also stand for the continuous mapping, $V : \omega := (s, p) \in \Omega \longmapsto V(\omega) := V(s) \in \mathbb{R}^J$.

Given the asset price, $q \in \mathbb{R}^J$, a portfolio, $z = (z_j) \in \mathbb{R}^J$, is a contract, which an agent may buy or sell at the cost of $q \cdot z$ units of account at $t = 0$, specifies the quantities, z_j, of each asset $j \in J$ (bought, if positive, or sold, if negative) and delivers a flow, $V(s) \cdot z$, of conditional payoffs, in every state, $s \in S$.

We recall from Cornet-De Boisdeffre (2009) that agents, having private beliefs and no clue of how prices are determined, may always infer a self-attainable arbitrage-free refinement of the anticipation structure, (Ω_i), or, equivalently, of the information structure, (S_i), along the following Definition. They may infer this refinement from observing a price, or mutually beneficial trade opportunities on the financial market. So, we now assume, at the outset, that $(\Omega_i) \in \mathcal{AS}$, is arbitrage-free.

Definition 1 *Given $q \in \mathbb{R}^J$, the anticipation structure, $(\Omega_i) \in \mathcal{AS}$, or, equivalently, the information structure, (S_i), is q-arbitrage-free if following Condition holds:*
(a) $\nexists (i,z) \in I \times \mathbb{R}^J : -q \cdot z \geqslant 0$ and $V(s) \cdot z \geqslant 0$, $\forall s \in S_i$, with one strict inequality.
The structure, (Ω_i) or (S_i), is arbitrage-free, if it is q-arbitrage-free for some $q \in \mathbb{R}^J$.

2.2 Agents' behaviours and the concept of equilibrium

Along Cornet-De Boisdeffre (2002), each agent, $i \in I$, receives an endowment, $e_i := (e_{is}) \in \mathbb{R}_+^{HS'_i}$, granting the commodity bundles, $e_{i0} \in \mathbb{R}_+^H$ at $t = 0$, and $e_{is} \in \mathbb{R}_+^H$, in each state $s \in S_i$, if this state prevails at $t = 1$.

Given the observed prices, $\omega_0 := (p_0, q) \in \mathbb{R}_+^H \times \mathbb{R}^J$, at $t = 0$, the generic i^{th} agent's consumption set is that of continuous mappings, $x : \Omega'_i \rightarrow \mathbb{R}_+^H$ (where $\Omega'_i := \{0\} \cup \Omega_i$):

$X_i := \mathcal{C}\ (\Omega'_i,\ \mathbb{R}_+^H)$.

Thus, her consumptions, $x \in X_i$, are mappings, relating $s = 0$ to a consumption decision, $x_{\omega_0} := x_0 := (x_0^h) \in \mathbb{R}_+^H$, at $t = 0$, and, continuously on Ω_i, every anticipation, $\omega := (s, p_s) \in \Omega_i$, to a consumption decision, $x_\omega := (x_\omega^h) \in \mathbb{R}_+^H$, at $t = 1$, which is conditional on the joint observation of state s, and price p_s, on the spot market, at $t = 1$. The generic i^{th} agent elects a strategy, $(x, z) \in X_i \times \mathbb{R}^J$, in her budget set, namely:

$$B_i(\omega_0) := \{(x, z) \in X_i \times \mathbb{R}^J :\ p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ \textit{and} \ \ p_s \cdot (x_\omega - e_{is}) \leqslant V(s) \cdot z,\ \forall \omega := (s, p_s) \in \Omega_i\}.$$

We notice that a consumption implementable in the budget set is bounded, from the definition of anticipation sets. Given agents' private beliefs, $(\pi_i) \in \Pi[(\Omega_i)]$, each consumer, $i \in I$, has preferences represented by the V.N.M. utility function:

$$\underline{x \in X_i \mapsto U_i^{\pi_i}(x) := \int_{\omega \in \Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega)} \ .$$

The above economy is denoted by $\mathcal{E} = \{(I, S, H, J), V, (\Omega_i)_{i \in I}, (e_i)_{i \in I}, (u_i)_{i \in I}\}$. It retains the standard small consumer price-taker hypothesis, along which no single agent's belief, or strategy, may, alone, have a significant impact on prices. It is said to be standard if, moreover, it meets the following Conditions:

- ***Assumption A1*** *(strong survival): for each* $i \in I$, $e_i \in \mathbb{R}^{HS'_i}_{++}$;
- ***Assumption A2****: for each* $i \in I$, u_i *is continuous, strictly concave and strictly increasing, that is, for every pair* $[(x, y), (x', y')] \in (\mathbb{R}^H_+ \times \mathbb{R}^H_+)^2$, *such that* $(x', y') \geqslant (x, y)$ *and* $(x', y') \neq (x, y)$, *the relation* $u_i(x, y) < u_i(x', y')$ *holds*;
- ***Assumption A3****: for every* $(i, h) \in I \times H$, *the mapping* $(x, y) \mapsto \partial u_i(x, y)/\partial y^h$ *is defined and continuous on* $\{(x, y) \in \mathbb{R}^H_+ \times \mathbb{R}^H_+ : y^h > 0\}$, *and* $(\inf_A \ \partial u_i(x, y)/\partial y^h) > 0$, *for every bounded subset* $A \subset \{(x, y) \in \mathbb{R}^H_+ \times \mathbb{R}^H_+ : y^h > 0\}$.

Remark 1 In Assumption *A2*, which could be weakened, strict concavity is retained to alleviate the proof of a selection amongst optimal strategies (see the proof of Lemma 2-(vi)-(vii) in the Appendix). The technical Assumption *A3* is consistent with the standard Inada Conditions, but does not require them.

The consumer's behaviour is to elect an optimal strategy within her buget set. With clearing market prices, this yields the following concept of equilibrium, which is both sequential, since all agents have self-fulfilling forecasts (under condition (a)), and temporary, since their anticipations, of the endogenous prices, are exogenous:

Definition 2 *A collection of prices and forecasts,* $\omega_0 := (p_0, q) \in \mathbb{R}^H_+ \times \mathbb{R}^J$ *and* $\omega_s = (s, p_s) \in \Omega$ *for each* $s \in \underline{\mathbf{S}}$*, of beliefs,* $\pi_i \in \Pi[(\Omega_i)]$*, and strategies,* $[(x_i, z_i)] \in \times_{i \in I} B_i(\omega_0)$*, is a sequential equilibrium of the economy* $\mathcal{E}$*, or correct foresight equilibrium (CFE), if the following Conditions hold:*

(a) $\forall s \in \underline{\mathbf{S}},\ \omega_s \in \cap_{i \in I} \Omega_i$*;*

(b) $\forall i \in I,\ (x_i, z_i) \in \arg\max_{(x,z) \in B_i(\omega_0)}\ U_i^{\pi_i}(x)$*;*

(c) $\sum_{i \in I} (x_{i\omega_s} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}'$*;*

(d) $\sum_{i \in I} z_i = 0$*.*

Under above conditions, each forecast, ω_s *(for* $s \in \underline{\mathbf{S}}$*), is said to support equilibrium.*

Remark 2 Whenever $\#\Omega_i = \#S_i$, for every $i \in I$, the above notion of equilibrium coincides with De Boisdeffre's (2007), where agents have perfect price foresight in every state that may prevail (i.e., $s \in \underline{\mathbf{S}}$).

3 The existence theorem

We now state our main Theorem, proved in Section 4, and present the minimum uncertainty that agents face on their forecasts, as a consequence of their unawareness of how prices are determined and of other agents' characteristics and beliefs.

3.1 An uncertainty principle and the existence of equilibrium

Definition 3 *Let* E *be the set of sequential equilibria (CFE) of the economy,* $\mathcal{E}$*. The minimum uncertainty set is that of forecasts, which, for some beliefs,* $(\pi_i) \in \Pi[(\Omega_i)]$*, support a CFE, namely:* $\Delta = \{\omega = (\underline{s}, p) \in \underline{\mathbf{S}} \times P : \exists \{(\omega_s), (\pi_i), [(x_i, z_i)]\} \in E,\ \omega_{\underline{s}} = \omega\}$*.*

We notice, from De Boisdeffre (2007) and Assumption *A2*, that the set Δ is non-empty. The following Theorem states an existence property of standard economies.

Theorem 1 *Let a standard economy, $\mathcal{E}$, and its minimum uncertainty set, Δ, be given. The following Assertions hold:*

(i) *there exists* $\varepsilon \in]0,1[$*, such that* $\Delta \subset \underline{\mathbf{S}} \times [\varepsilon, 1]^H$*;*

(ii) *if* $\Delta \subset \cap_{i \in I} \Omega_i$*, equilibrium exists and any beliefs,* $(\pi_i) \in \Pi[(\Omega_i)]$*, support a CFE.*

Before proving Assertion (ii) in the next Section, we prove Assertion (i) hereafter.

Proof of Assertion (i) Let a standard economy, $\mathcal{E}$, and a forecast, $\omega := (s,p) \in \Delta$, be given, which supports a CFE, $\{(\omega_s)_{s \in \underline{\mathbf{S}}'}, (\pi_i), [(x_i, z_i)]\}$. The relation $p := (p^h)_{h \in H} \in \mathbb{R}^H_{++}$ is standard from Assumption *A2* and Definition 2-(b).

Let $m := (\min_{(i,s,h) \in I \times \underline{\mathbf{S}} \times H} \ e^h_{is}) \in \mathbb{R}_{++}$ and $M := (\max_{(s,h) \in \underline{\mathbf{S}}' \times H} \ \sum_{i \in I} e^h_{is}) \in \mathbb{R}_{++}$ be given, along Assumption *A1*. Then, the relations $(x_{i0}) \geqslant 0$, $(x_{i\omega}) \geqslant 0$, $\sum_{i \in I}(x_{i0} - e_{i0}) = 0$ and $\sum_{i \in I}(x_{i\omega} - e_{is}) = 0$, which hold from Definition 2-(c), yield $x_{i0} \in [0,M]^H$ and $x_{i\omega} \in [0,M]^H$, for each $i \in I$.

Let $\alpha := \inf \ \ \partial u_i(x,y)/\partial y^h$, for every $(i,h,(x,y)) \in I \times H \times \{(x,y) \in [0,M]^{2H} : \ y^h > 0\}$, and $\beta := \max \ \ \partial u_i(x,y)/\partial y^h$, for every $(i,h,(x,y)) \in I \times H \times \{(x,y) \in [0,M]^{2H} : \ y^h \geqslant m \ \}$, and $\gamma = \beta/\alpha$ be strictly postive numbers, along Assumption *A3*, above.

Let $(h,h') \in H^2$ be given and assume, by contraposition, that $p^h/p^{h'} > \gamma$. From the above relations, there exists at least one agent, say $i = 1$, unwilling to sell good h, when forecasting $\omega := (s,p) \in \Delta$, that is, $x^h_{1\omega} \in [m,M]$. We let the reader check, as tedious and standard, that agent $i = 1$, starting from (x_1, z_1), could find a utility increasing strategy, $(x^*_1, z_1) \in B_1(\omega_0)$, modifying her consumptions in her forecast ω only, such that $x^{*h}_{1\omega} < x^h_{1\omega}$ and $x^{*h'}_{1\omega} > x^{h'}_{1\omega}$. Indeed, with $p^h_s/p^{h'}_s > \gamma$, she has an incentive to sell a small amount of the expensive commodity h in exchange for commodity h'. Hence, (x_1, z_1) cannot be an equilibrium strategy. This contradiction proves the relation $p^h/p^{h'} \leqslant \gamma$. We let the reader check, from the above relations, $p \in \mathbb{R}^H_{++}$, $\|p\| = 1$, $p^h/p^{h'} \leqslant \gamma$ (for each $(h,h') \in H^2$), that $p^h \geqslant \varepsilon := 1/\gamma \# H$ holds for each $h \in H$. □

3.2 The Theorem's Condition

Under the Theorem's Condition, $\Delta \subset \cap_{i\in I}\Omega_i$, a CFE exists, for any beliefs, $(\pi_i) \in \Pi[(\Omega_i)]$. We now explain why Δ is called a set of "*minimum uncertainty*" and argue how agents might, or should, include it into their anticipation sets.

On the former issue, when the structure of beliefs, today, is unknown to consumers, no equilibrium price should be ruled out a priori. Theoretically, the set, Δ, of all possible equilibrium prices tomorrow (along some beliefs today), is one of incompressible uncertainty. Practically, it could be so, in times of enhanced uncertainty, volatility or erratic change in beliefs, which prevent coordination from agents or institutions. No equilibrium forecast of Δ might, then, be ruled out.

On the latter issue, the model specifies *normalised* prices. It is often possible to observe past prices and reckon their *relative* values, in a wide array of situations, or states, which typically replicate over time. For example, the price of many assets can be followed daily over decades, hence, in the daily state. Long series also exist for consumption goods. In time series, relative prices would vary between observable (upper and lower) bounds in the various states. Along a sensible assumption, provided series be long enough, all CFE forecasts (i.e., those of Δ) should lie within these boundaries, and the latter could serve to construct an embedding of Δ. Such a statistical method requires no demanding model or awareness of the primitives of the economy. It could be implemented by a financial institution or intermediary.[3]

[3] E.g., if the future reflects the past, if $\underline{\mathbf{S}}$ is also a set of past states and, for every $s \in \underline{\mathbf{S}}$, the past price serie, $(p^t_s) \in (P)^{T_s}$ (where $T_s \in \mathbb{N}$) is large, the set $\{(s, y_s) \in \underline{\mathbf{S}} \times P : \ y_s = \sum_{t=1}^{T_s} \alpha_t p^t_s \ / \| \sum_{t=1}^{T_s} \alpha_t p^t_s \|, \ (\alpha_t) \in \mathbb{R}^{T_s}_+, \ \sum_{t=1}^{T_s} \alpha_t{=}1\}$, could easily be checked, iteratively, to contain self-fulfilling forecasts, hence, estimated to contain Δ.

Even if Δ (or a bigger set) were public, individuals might have idiosyncratic uncertainty, along their incomplete information or personal feelings. Then, anticipations would not be symmetric. Moreover, on actual markets, even if agents had symmetric anticipations, their supporting beliefs would typically remain private and determine future prices. Then, no agent, or institution, could spot a self-fulfilling forecast precisely within Δ. Only sets of possible prices (not a single one) could be anticipated on spot markets. Forecasts would obey an "*uncertainty principle*".

If the condition, $\Delta \subset \cap_{i \in I} \Omega_i$, holds, the Theorem shows that equilibrium may always be reached, e.g., by tatonnement. This remains true if beliefs, $(\pi_i) \in \Pi[(\Omega_i)]$, change ex ante. With no price model, agents cannot refine their anticipations beyond the unique arbitrage-free anticipation structure (here confounded with $(\Omega_i) \in \mathcal{AS}$), which they infer from markets, along Cornet-De Boisdeffre (2009). This unique structure only depends on assets' payoffs and agents' prior forecasts (and on no other primitive). In the path to equilibrium, this anticipation structure cannot change, but its supporting beliefs may change. Such changes could affect equilibrium prices and allocation only, not anticipations. This approach of the sequential equilibrium (and path to it) requires no particular knowledge or computation from consumers, and seems to picture actual behaviours on markets, where agents make exogenous forecasts under uncertainty, seek to take advantage of arbitrage opportunities and learn from them, and where market play drives prices, demands and beliefs to equilibrium. This approach drops Radner's (1972 & 1979) rational expectations.

4 The existence proof

Throughout, we set as given arbitrary beliefs, $(\pi_i) \in \Pi[(\Omega_i)]$, assume the economy, $\mathcal{E}$, is standard and that the Theorem's condition, $\Delta \subset \cap_{i \in I} \Omega_i$, holds.

The proof's principle is to construct a sequence of auxiliary economies, with finite anticipation sets, refining and tending to the initial sets, (Ω_i). Each finite economy admits an equilibrium, which we set as given, along De Boisdeffre (2007). Then, we derive from the sequence of finite equilibria an equilibrium of the initial economy, $\mathcal{E}$. The auxiliary economies build on partitions of (Ω_i), presented hereafter.

4.1 Finite partitions of agents' anticipation sets

We set as given $i \in I$, recall the second Section's definition of the anticipation set $\Omega_i := \cup_{s \in S_i} \{s\} \times P_s^i$ (a closed subset of $S \times P$), and let $K^n := (\mathbb{N} \cap [1, 2^n])^H$, for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, each $s \in S_i$ and each $k_n := (k_n^h) \in K^n$, we define the set $\Omega_i^{(s,k_n)} := \{s\} \times (P_s^i \cap \times_{h \in H}]\frac{k_n^h - 1}{2^n}, \frac{k_n^h}{2^n}])$, and let $K_s^n := \{k_n \in K^n : \pi_i(\Omega_i^{(s,k_n)}) > 0\}$. The above sets yield ever finer partitions, $\mathcal{P}_i^n := \{\Omega_i^{(s,k_n)}\}_{(s,k_n) \in S_i \times K_s^n}$, of Ω_i, for $n \in \mathbb{N}$.

Then, for every triple $(n, s, k_n) \in \mathbb{N} \times S_i \times K_s^n$, we set as given one unique element, $\omega_i^{(s,k_n)} \in \Omega_i^{(s,k_n)}$, and construct $\Omega_i^n := \{\omega_i^{(s,k_n)}\}_{(s,k_n) \in S_i \times K_s^n}$ such that $\Omega_i^n \subset \Omega_i^{n+1}$, for each $n \in \mathbb{N}$ (which is always possible). We define mappings, $\Phi_i^n : \Omega_i \to \Omega_i^n$, by $\Phi_i^n(\omega) := \omega_i^{(s,k_n)}$, for every tuple $(n, s, k_n, \omega) \in \mathbb{N} \times S_i \times K_s^n \times \Omega_i^{(s,k_n)}$, and probabilities, π_i^n, on Ω_i^n, by the relations $\pi_i^n(\omega_i^{(s,k_n)}) := \pi_i(\Omega_i^{(s,k_n)}) > 0$, for every triple $(n, s, k_n) \in \mathbb{N} \times S_i \times K_s^n$.

Throughout, the sequences, $\{\mathcal{P}_i^n\}_{n \in \mathbb{N}}$, $\{\Omega_i^n\}_{n \in \mathbb{N}}$, $\{\Phi_i^n\}_{n \in \mathbb{N}}$ and $\{\pi_i^n\}_{n \in \mathbb{N}}$, respectively, of partitions, sub-sets, mappings and probabilities on Ω_i, are defined as above for each $i \in I$, and meet the following properties:

Lemma 1 *For each $i \in I$, the above sequences, $\{\Omega_i^n\}_{n \in \mathbb{N}}$ & $\{\Phi_i^n\}_{n \in \mathbb{N}}$, are such that:*

(i) $\Omega_i = \overline{\lim_{n \to \infty} \nearrow \Omega_i^n} = \overline{\cup_{n \in \mathbb{N}} \Omega_i^n}$, that is, $\cup_{n \in \mathbb{N}} \Omega_i^n$ is dense in Ω_i;

(ii) for every $\omega \in \Omega_i$, $\omega = \lim_{n \to \infty} \Phi_i^n(\omega)$, and $\{\Phi_i^n(\omega)\}$ converges uniformly to ω;

(iii) $(\Omega_i^n) \in \mathcal{AS}$ is arbitrage-free, for every $n \in \mathbb{N}$.

Proof Assertions (i) and (ii) are straightforward from the definitions of $\{\Omega_i^n\}_{n\in\mathbb{N}}$ and $\{\Phi_i^n\}_{n\in\mathbb{N}}$ and Assertion (iii) results immediately from Definition 1, the relations $S_i = \{s \in S : \exists p \in P, (s,p) \in \Omega_i^n\}$, which hold for evey pair $(i,n) \in I \times \mathbb{N}$, and the fact that the information structure, (S_i), is arbitrage-free. $\square$

4.2 The auxiliary economies, $\mathcal{E}^n$

Throughout, we let $n \in \mathbb{N}$ be given. We define a formal auxiliary economy, $\mathcal{E}^n$, with two periods, $t \in \{0,1\}$, finitely many agents $i \in I$, who receive the same endowments, $e_i \in \mathbb{R}_+^{HS_i'}$, consume and exchange the same goods on spot markets, $h \in H$, and trade the same type of assets, $j \in J$, as in Section 2. This formal economy is of the type described in De Boisdeffre (2007) and presented hereafter.

For each $i \in I$, we let $\widetilde{\Omega}_i^n := \{i\} \times \Omega_i^n$, $\Theta_i^n := \underline{\mathbf{S}} \cup \widetilde{\Omega}_i^n$ and $\Theta^n := \cup_{i\in I} \Theta_i^n$ be given sets, and define, from Section 2, a $\Theta^n \times J$ payoff matrix, V^n, by $V^n(s) := V(s) \in \mathbb{R}^J$, for every $s \in \underline{\mathbf{S}}$, and $V^n((i,\omega)) := V(\omega) := V(s) \in \mathbb{R}^J$, for every $i \in I$ and every $\omega := (s,p) \in \Omega_i^n$. The economy's, $\mathcal{E}^n$, state space, information structure, pooled information set and payoff matrix are, respectively Θ^n, (Θ_i^n), $\underline{\mathbf{S}}$ and V^n, which make the financial and information structure, $[V^n, (\Theta_i^n)]$, arbitrage-free, along Definition 1, above.

In each (realizable) state $s \in \underline{\mathbf{S}}$, the generic i^{th} agent is assumed to anticipate with certainty and perfect foresight the future spot price (say $p_s^n \in \mathbb{R}_+^H$) of commodities, which is endogenous. In each other state, $(i,s,p) \in \widetilde{\Omega}_i^n$ - a state which is unrealizable and purely formal, the agent expects with certainty the spot price $p \in \mathbb{R}_+^H$ to prevail.

Along this specification, the economy, $\mathcal{E}^n$, is (up to a slight change in notations) of the De Boisdeffre's (2007) type. Given the market prices, $p := (p_s) \in \mathbb{R}_+^{H\underline{\mathbf{S}}'}$ and $q \in \mathbb{R}^J$, which are either observed or perfectly anticipated, the generic i^{th} agent's consumption set, X_i^n, budget set, $B_i^n(p,q)$, utility function, u_i^n, and the concept of equilibrium are defined as follows, respectively:

$\underline{X_i^n := \mathbb{R}_+^{H\underline{\mathbf{S}}'} \times \mathbb{R}_+^{H\Omega_i^n}}$, whose generic element is $x := ((x_s)_{s\in\underline{\mathbf{S}}'}, (x_\omega)_{\omega\in\Omega_i^n})$;

$$\underline{B_i^n(p,q)} := \{\ (x,z) \in X_i^n \times \mathbb{R}^J :\ p_0\cdot(x_0 - e_{i0}) \leqslant -q\cdot z \ \ \textit{and}\ \ p_s\cdot(x_s - e_{is}) \leqslant V(s)\cdot z,\ \forall s \in \underline{\mathbf{S}}$$
$$\textit{and}\ \ p\cdot(x_\omega - e_{is}) \leqslant V(s)\cdot z,\ \forall \omega := (s,p) \in \Omega_i^n\ \};$$

$$x \in X_i^n \mapsto \underline{u_i^n(x) := \sum_{s\in\underline{\mathbf{S}}} \frac{u_i(x_0,x_s)}{2^{n+1}\#\underline{\mathbf{S}}} + (1 - \frac{1}{2^{n+1}}) \sum_{\omega\in\Omega_i^n} u_i(x_0,x_\omega)\pi_i^n(\omega)}.$$

Definition 4 *A collection of prices,* $(p^n,q^n) \in \mathbb{R}_+^{H\underline{\mathbf{S}}'} \times \mathbb{R}^J$ *and strategies,* $[(x_i^n,z_i^n)] \in \times_{i\in I} B_i^n(p^n,q^n)$*, is an equilibrium of the economy* $\mathcal{E}^n$ *if the following Conditions hold:*

(a) $\forall i \in I,\ (x_i^n,z_i^n) \in \arg\max_{(x,z)\in B_i^n(p^n,q^n)}\ u_i^n(x)$*;*

(b) $\sum_{i\in I}(x_{is}^n - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}'}$*;*

(c) $\sum_{i\in I} z_i^n = 0$.

From De Boisdeffre's (2007) Theorem 1 and its proof, the economy, $\mathcal{E}^n$, admits an equilibrium, $\mathcal{C}^n := ((p^n,q^n),[(x_i^n,z_i^n)]) \in (\mathbb{R}_+^{H\underline{\mathbf{S}}'} \times \mathbb{R}^J) \times (\times_{i\in I}\ B_i^n(p^n,q^n))$, such that $\|p_s^n\| = 1$, for each $s \in \underline{\mathbf{S}}$, and $\|p_0^n\| + \|q^n\| = 1$. It meets the following properties.

Lemma 2 *Let a sequence of equilibria,* $\{\mathcal{C}^n\}_{n\in\mathbb{N}}$*, be defined from above and, for every* $i \in I$*,* $z \in \mathbb{R}^J$*, and* $\omega = (s,p) \in \Omega$*, let* $B_i(\omega,z) := \{\ x \in \mathbb{R}_+^H : p\cdot(x - e_{is}) \leqslant V(s)\cdot z\ \}$ *be a given set of consumptions. Then, the following Assertions hold*:

(i) $\forall (n,i,s) \in \mathbb{N} \times I \times \underline{\mathbf{S}}'$*,* $x_{is}^n \in [0,E]^H$ *where* $E := \max_{(s,h)\in S'\times H} \sum_{i\in I} e_{is}^h$*;*

(ii) $\forall s \in \underline{\mathbf{S}},\ (s,p_s^n) \in \Delta \subset \cap_{i\in I}\Omega_i$*;*

(iii) it may be assumed to exist $q^* = \lim_{n\to\infty}\ q^n$ *&* $p_s^* = \lim_{n\to\infty}\ p_s^n$*, for each* $s \in \underline{\mathbf{S}}'$*; we let* $\omega_0^* := (p_0^*,q^*) \in \mathbb{R}_+^H \times \mathbb{R}^J$ *satisfy* $\|\omega_0^*\| = 1$ *and* $\{\omega_s^* := (s,p_s^*)\}_{s\in\underline{\mathbf{S}}} \subset (\cap_{i\in I}\Omega_i)$*;*

(iv) for each $s \in \underline{\mathbf{S}}'$*, it may be assumed to exist* $(x_{is}^*) := \lim_{n\to\infty}\ (x_{is}^n)_{i\in I} \in (\mathbb{R}^H)^I$*, such that* $\sum_{i\in I} (x_{is}^* - e_{is}) = 0$*, and we let* $(x_{i\omega_s^*}^*) := (x_{is}^*)$*;*

(v) *it may be assumed to exist* $(z_i^*) = \lim_{n\to\infty} (z_i^n)$, *such that* $\sum_{i\in I} z_i^* = 0$;

(vi) $\forall (i,s) \in I \times \underline{\mathbf{S}}$, $\{x^*_{i\omega^*_s}\} = \arg\max\ u_i(x^*_{i\omega^*_0}, x)$, *for* $x \in B_i(\omega^*_s, z^*_i)$, *defined from above;*

(vii) *for each* $i \in I$, *the correspondence* $\omega \in \Omega_i \mapsto \arg\max\ u_i(x^*_{i\omega^*_0}, x)$, *for* $x \in B_i(\omega, z^*_i)$, *is a continuous mapping, denoted by* $\omega \mapsto x^*_{i\omega}$, *and the mapping,* $x^*_i : \omega \in \Omega'_i \mapsto x^*_{i\omega}$, *defined from above, is a consumption plan, that is,* $x^*_i \in X_i$;

$(viii)$ *for each* $i \in I$, $U_i^{\pi_i}(x^*_i) = \lim_{n\to\infty} u^n_i(x^n_i)$, *as defined from above.*

Proof see the Appendix. □

4.3 An equilibrium of the initial economy

We now prove Assertion (ii) of Theorem 1, via the following Claim.

Claim 1 *The collection of prices and forecasts,* (ω^*_s), *beliefs,* (π_i), *allocation,* (x^*_i), *and portfolios,* (z^*_i), *of Lemma 2, defines a C.F.E. of the economy* $\mathcal{E}$.

Proof Let us define $\mathcal{C}^* := ((\omega^*_s), (\pi_i), [(x^*_i, z^*_i)])$ as in Claim 1. From Lemma 2-(ii)-(iii)-(iv)-(v), $\mathcal{C}^*$ meets Conditions (a)-(c)-(d) of Definition 2 of equilibrium above. Hence, it suffices to show that both relations $[(x^*_i, z^*_i)] \in \times_{i\in I} B_i(\omega^*_0)$ and Definition 2-(b) hold.

First, we set $i \in I$ as given, and show: $(x^*_i, z^*_i) \in B_i(\omega^*_0)$. From the definition of $\mathcal{C}^n$, the relations $p^n_0 \cdot (x^n_{i0} - e_{i0}) \leqslant -q^n \cdot z^n_i$ hold, for each $n \in \mathbb{N}$, and, yield $p^*_0 \cdot (x^*_{i0} - e_{i0}) \leqslant -q^* \cdot z^*_i$, in the limit. From Lemma 2-(vii), the relations $x^*_i \in X_i$ and $p_s \cdot (x^*_{i\omega} - e_{is}) \leqslant V(s) \cdot z^*_i$ also hold, for every $\omega = (s, p_s) \in \Omega_i$. Hence, $(x^*_i, z^*_i) \in B_i(\omega^*_0)$ holds, for each $i \in I$.

Next, we assume, by contraposition, that $\mathcal{C}^*$ fails to meet Condition (b) of Definition 2, that is, there exist $i \in I$, $(x, z) \in B_i(\omega^*_0)$ and $\varepsilon \in \mathbb{R}_{++}$, such that:

$$(I) \qquad \varepsilon + U_i^{\pi_i}(x^*_i) < U_i^{\pi_i}(x).$$

We may assume:

$(II)\qquad \exists\ (\delta, M) \in \mathbb{R}^2_{++}\colon\ x_\omega \in [\delta, M]^H,\ \forall \omega \in \Omega_i.$

The existence of an upper bound to consumptions x_ω (for $\omega \in \Omega_i$) results from the relation $(x, z) \in B_i(\omega_0^*)$, which implies a bound to financial transfers, and the fact that Ω_i is closed in $S \times P$. Moreover, for $\alpha \in]0, 1]$ small enough, the strategy $(x^\alpha, z^\alpha) := ((1-\alpha)x + \alpha e_i, (1-\alpha)z) \in B_i(\omega_0^*)$ meets both relations (I) and (II), from Assumption *A1*and the uniform continuity (on a compact set) of the mapping $(\alpha, \omega) \in [0, 1] \times \Omega_i \mapsto u_i(x_0^\alpha, x_\omega^\alpha)$. So, relations (II) may be assumed.

Then, we let the reader check, as immediate from the relations (I)-(II) and $(x, z) \in B_i(\omega_0^*)$, from Lemma 2-$(iii)$, the definition of Ω_i, Assumptions *A1-A2* and continuity arguments, that we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

$(III)\quad p_0^* \cdot (x_0 - e_{i0}) \leqslant -q^* \cdot z$ and $p_s \cdot (x_\omega - e_{is}) \leqslant -\gamma + V(s) \cdot z,\ \forall \omega := (s, p_s) \in \Omega_i.$

From relations (I)-(II)-(III), we may also assume there exists $\gamma' \in]0, \gamma[$, such that:

$(IV)\quad p_0^* \cdot (x_0 - e_{i0}) \leqslant -\gamma' - q^* \cdot z$ and $p_s \cdot (x_\omega - e_{is}) \leqslant -\gamma' + V(s) \cdot z,\ \forall \omega := (s, p_s) \in \Omega_i.$

Indeed, the above assertion is obvious, from relations (III), if $p_0^* \cdot (x_0 - e_{i0}) < -q^* \cdot z$. Assume that $p_0^* \cdot (x_0 - e_{i0}) = -q^* \cdot z$. If $p_0^* = 0$, then, $q^* \neq 0$, from Lemma 2-(iii), and relations (IV) hold if we replace z by $z - q^*/N$, for $N \in \mathbb{N}$ big enough. If $p_0^* \neq 0$ and $x_0 \neq 0$, the desired assertion results from Assumption *A1* and above. Otherwise, $-q^* \cdot z = -p_0^* \cdot e_{i0} < 0$, and a slight change in portfolio insures relations (IV).

From relations (IV), the continuity of the scalar product and Lemma 2-(ii)-(iii), there exists $N_1 \in \mathbb{N}$, such that, for every $n \geqslant N_1$:

$$(V)\quad \left[\begin{array}{l} p_0^n \cdot (x_0 - e_{i0}) \leqslant -q^n \cdot z \\ p_s^n \cdot (x_{\omega_s^*} - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_\omega - e_{is}) \leqslant V(s) \cdot z,\ \forall \omega := (s, p_s) \in \Omega_i^n \end{array}\right. .$$

Along relations (V), for each $n \geqslant N_1$, we define, in the economy $\mathcal{E}^n$, the strategy $(x^n, z) \in B_i^n(p^n, q^n)$ by: $x_0^n := x_0$, $x_s^n := x_{\omega_s^*}$, for every $s \in \underline{\mathbf{S}}$, and $x_{(i,\omega)}^n := x_\omega$, for every $\omega \in \Omega_i^n$. We recall the following definitions:

- $U_i^{\pi_i}(x) := \int_{\omega \in \Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega)$;
- $u_i^n(x^n) := \sum_{s \in \underline{\mathbf{S}}} \frac{u_i(x_0, x_s^n)}{2^{n+1} \# \underline{\mathbf{S}}} + (1 - \frac{1}{2^{n+1}}) \sum_{\omega \in \Omega_i^n} u_i(x_0, x_\omega) \pi_i^n(\omega)$.

Then, from above, relation (II), Lemma 1-(ii), and the uniform continuity of $x \in X_i$ and u_i on compact sets, there exists $N_2 \geqslant N_1$ such that:

(VI) $\quad |U_i^{\pi_i}(x)\text{-}u_i^n(x^n)| < \int_{\omega \in \Omega_i} |u_i(x_0, x_\omega)\text{-}u_i(x_0, x_{\Phi_i^n(\omega)})| d\pi_i(\omega) + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$, for every $n \geqslant N_2$.

From equilibrium conditions and Lemma 2-$(viii)$, there exists $N_3 \geqslant N_2$, such that:

(VII) $\quad u_i^n(x^n) \leqslant u_i^n(x_i^n) < \frac{\varepsilon}{2} + U_i^{\pi_i}(x_i^*)$, for every $n \geqslant N_3$.

Let $n \geqslant N_3$ be given. The above Conditions (I)-(VI)-(VII) yield, jointly:

$U_i^{\pi_i}(x) < \frac{\varepsilon}{2} + u_i^n(x^n) \leqslant \frac{\varepsilon}{2} + u_i^n(x_i^n) < \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x)$.

This contradiction proves that $\mathcal{C}^*$ is indeed a C.F.E. and Theorem 1-(ii) holds. □

Appendix: proof of Lemma 2

Lemma 2 *Let a sequence of equilibria,* $\{\mathcal{C}^n\}_{n \in \mathbb{N}}$*, be defined from above and, for every* $i \in I$*,* $z \in \mathbb{R}^J$*, and* $\omega = (s, p) \in \Omega$*, let* $B_i(\omega, z) := \{\ x \in \mathbb{R}_+^H : p \cdot (x - e_{is}) \leqslant V(s) \cdot z\ \}$ *be a given set of consumptions. Then, the following Assertions hold*:

(i) $\forall (n, i, s) \in \mathbb{N} \times I \times \underline{\mathbf{S}}'$, $x_{is}^n \in [0, E]^H$ *where* $E := \max_{(s,h) \in S' \times H} \sum_{i \in I} e_{is}^h$*;*

(ii) $\forall s \in \underline{\mathbf{S}}$, $(s, p_s^n) \in \Delta \subset \cap_{i \in I} \Omega_i$*;*

(iii) it may be assumed to exist $q^* = \lim_{n\to\infty}\ q^n$ *& * $p^*_s = \lim_{n\to\infty}\ p^n_s$*, for each* $s \in \underline{\mathbf{S}}'$*; we let* $\omega^*_0 := (p^*_0, q^*) \in \mathbb{R}^H_+ \times \mathbb{R}^J$ *satisfy* $\|\omega^*_0\| = 1$ *and* $\{\omega^*_s := (s, p^*_s)\}_{s\in\underline{\mathbf{S}}} \subset (\cap_{i\in I}\Omega_i)$*;*

(iv) for each $s \in \underline{\mathbf{S}}'$*, it may be assumed to exist* $(x^*_{is}) := \lim_{n\to\infty}\ (x^n_{is})_{i\in I} \in (\mathbb{R}^H)^I$*, such that* $\sum_{i\in I} (x^*_{is} - e_{is}) = 0$*, and we let* $(x^*_{i\omega^*_s}) := (x^*_{is})$*;*

(v) it may be assumed to exist $(z^*_i) = \lim_{n\to\infty}\ (z^n_i)$*, such that* $\sum_{i\in I} z^*_i = 0$*;*

(vi) $\forall (i,s) \in I \times \underline{\mathbf{S}}$*,* $\{x^*_{i\omega^*_s}\} = \arg\max\ u_i(x^*_{i\omega^*_0}, x)$*, for* $x \in B_i(\omega^*_s, z^*_i)$*, defined from above;*

(vii) for each $i \in I$*, the correspondence* $\omega \in \Omega_i \mapsto \arg\max\ u_i(x^*_{i\omega^*_0}, x)$*, for* $x \in B_i(\omega, z^*_i)$*, is a continuous mapping, denoted by* $\omega \mapsto x^*_{i\omega}$*, and the mapping,* $x^*_i : \omega \in \Omega'_i \mapsto x^*_{i\omega}$*, defined from above, is a consumption plan, that is,* $x^*_i \in X_i$*;*

(viii) for each $i \in I$*,* $U^{\pi_i}_i(x^*_i) = \lim_{n\to\infty} u^n_i(x^n_i)$*, as defined from above.*

Proof

Assertion (i) The relations $(x^n_{is}) \geqslant 0$ and $\sum_{i\in I}(x^n_{is} - e_{is}) = 0$ hold from the definition of $\mathcal{C}^n$, and yield $x^n_{is} \in [0, E]^H$, for each $(i, s, n) \in I \times \underline{\mathbf{S}}' \times \mathbb{N}$, where $E := \max_{(s,h)\in S'\times H} \sum_{i\in I} e^h_{is}$. □

Assertion (ii) is immediate from the definition of Δ and construction of $\mathcal{E}^n$ and $\mathcal{C}^n$. □

Assertion (iii) The limits, $q^* = \lim_{n\to\infty}\ q^n$ and $p^*_s = \lim_{n\to\infty}\ p^n_s$, for each $s \in \underline{\mathbf{S}}'$, may be assumed to exist from compactness arguments (all sequences being bounded). The relations, $\omega^n_0 \in \mathbb{R}^H_+ \times \mathbb{R}^J$, $\|\omega^n_0\| = 1$ and $\{\omega^n_s\}_{s\in\underline{\mathbf{S}}} \subset (\cap_{i\in I}\Omega_i)$, which hold for each $n \in \mathbb{N}$ (from the construction of $\mathcal{C}^n$ under the above Definition 4), pass to limit and yield $\omega^*_0 \in \mathbb{R}^H_+ \times \mathbb{R}^J$, $\|\omega^*_0\| = 1$ and $\{\omega^*_s\}_{s\in\underline{\mathbf{S}}} \subset (\cap_{i\in I}\Omega_i)$, a closed set. □

Assertion (iv), by the same token, results immediately from Assertion (i) and the same compactness and closedness arguments as above. □

Assertion (v) By the same token, it suffices to bound the sequence, $\{(z_i^n)\}_{n\in\mathbb{N}}$. We let $\delta := \max_{i\in I} \|e_i\|$ and consider the vector spaces, $Z_i := \{z \in \mathbb{R}^J : V(s)\cdot z = 0, \forall s \in S_i\}$ and its orthogonal, $Z_i^\perp$ (for each $i \in I$), and $Z := \sum_{i\in I} Z_i$. The definition of $\{\mathcal{C}^n\}_{n\in\mathbb{N}}$ yields, from budget constraints and market clearing conditions:

(I) $\qquad [\sum_{i\in I}\ z_i^n = 0 \ \textit{and}\ V(s)\cdot z_i^n \geqslant -\delta,\ \forall(i,s) \in I\times S_i]$, for every $n \in \mathbb{N}$.

For each $(i,n) \in I \times \mathbb{N}$, let $z_i^n := z_i^{on} \oplus z_i^{\perp n}$ be the orthogonal decomposition of z_i^n on $Z_i \times Z_i^\perp$. Condition (I) above is also written:

(II) $\qquad [\sum_{i\in I}\ z_i^{\perp n} = -\sum_{i\in I}\ z_i^{on} \in Z \ \textit{and}\ V(s)\cdot z_i^{\perp n} \geqslant -\delta,\ \forall(i,s) \in I\times S_i]$, for all $n \in \mathbb{N}$.

We show, first, that the sequence, $\{(z_i^{\perp n})\}_{n\in\mathbb{N}}$ is bounded. If not, there exists an extracted sequence, $\{(z_i^{\perp\varphi(n)})\}$, such that $n < \|(z_i^{\perp\varphi(n)})\| \leqslant n+1$, for any $n \in \mathbb{N}$. Then, the portfolios $(\overline{z}_i^n) := \frac{1}{n}(z_i^{\perp\varphi(n)})$ meet the relations $1 < \|(\overline{z}_i^n)\| \leqslant 1+\frac{1}{n}$, for all $n \in \mathbb{N}$, and:

(III) $\qquad \sum_{i\in I}\ \overline{z}_i^n \in Z \ \textit{and}\ V(\omega_i)\cdot\overline{z}_i^n \geqslant -\frac{\delta}{n},\ \forall(i,s) \in I\times S_i.$

The sequence $\{(\overline{z}_i^n)\}$ may be assumed to converge to (z_i^*), such that $\|(z_i^*)\|= 1$ and:

(IV) $\qquad \sum_{i\in I} z_i^* \in Z \ \textit{and}\ V(\omega_i)\cdot z_i^* \geqslant 0,\ \forall(i,s) \in I\times S_i.$

From relation (IV) and an immediate corollary of Cornet-De Boisdeffre (2002, Proposition 3.1, p. 401), the relation $(z_i^*) \in (\times_{i\in I} Z_i^\perp) \cap (\times_{i\in I} Z_i^o) = \{0\}$ holds, which contradicts the above, $\|(z_i^*)\| = 1$. Hence, the sequence $\{(z_i^{\perp n})\}$ is bounded, say, for some $A \in \mathbb{R}_{++}$, the relation $\|(z_i^{\perp n})\| \leqslant A$ holds, for every $n \in \mathbb{N}$. Then, from the above relations, $\sum_{i\in I} z_i^n := \sum_{i\in I} z_i^{on} \oplus \sum_{i\in I} z_i^{\perp n} = 0$, which hold for every $n \in \mathbb{N}$, it may be assumed that $\|(z_i^{on})\| \leqslant A$ and that $\|(z_i^n)\| \leqslant 2A$ also hold, for every $n \in \mathbb{N}$. Since the portfolio sequence $\{(z_i^n)\}_{n\in\mathbb{N}}$ is bounded, it may be assumed to converge, say to (z_i^*), which satisfies $\sum_{i\in I} z_i^* = 0$, since the relation $\sum_{i\in I} z_i^n = 0$ holds, for every $n \in \mathbb{N}$. $\square$

Assertion *(vi)* Let $(i,s) \in I \times \underline{\mathbf{S}}$ be given. For each $n \in \mathbb{N}$, the fact that $\mathcal{C}^n$ is an equilibrium of $\mathcal{E}^n$ implies: $x_{is}^n \in \arg\max_{y \in B_i(\omega_s^n, z_i^n)} u_i(x_{i0}^n, y)$.

As a standard application of Berge's Theorem (see, e.g., Debreu, 1959, p. 19), the correspondence (mapping from Assumption *A2)*, $(x,\omega,z) \in \mathbb{R}_+^H \times \Omega \times \mathbb{R}^J \mapsto \arg\max_{y \in B_i(\omega,z)} \; u_i(x,y)$, is continuous at $(x_{i0}^*, \omega_s^*, z_i^*)$, since u_i and B_i are. Moreover, from above, $(x_{i0}^*, x_{is}^*, \omega_s^*, z_i^*) = \lim_{n\to\infty}(x_{i0}^n, x_{is}^n, \omega_s^n, z_i^n)$. Hence, the latter relations (for every $n \in \mathbb{N}$) pass to that limit and yield the desired property, namely: $\{x_{i\omega_s^*}^*\} := \{x_{is}^*\} = \arg\max_{y \in B_i(\omega_s^*, z_i^*)} u_i(x_{i0}^*, y)$. □

Assertion *(vii)* Let $i \in I$ be given. For every $(\omega, n) \in \Omega_i \times \mathbb{N}$, the fact that $\mathcal{C}^n$ is an equilibrium of $\mathcal{E}^n$ and Assumption *A2* imply: $\{x_{i\Phi_i^n(\omega)}^n\} = \arg\max_{y \in B_i(\Phi_i^n(\omega), z_i^n)} u_i(x_{i0}^n, y)$.

By the same token as above, the mapping, $(x,\omega,z) \in \mathbb{R}_+^H \times \Omega \times \mathbb{R}^J \mapsto \arg\max_{y \in B_i(\omega,z)} \; u_i(x,y)$, is continuous. Moreover, from above, the relation $(x_{i0}^*, \omega, z_i^*) = \lim_{n\to\infty}(x_{i0}^n, \Phi_i^n(\omega), z_i^n)$ holds. Hence, the above relations (for every $n \in \mathbb{N}$) pass to that limit and yield a continuous mapping, $\omega \in \Omega_i \mapsto x_{i\omega}^* := \arg\max_{y \in B_i(\omega, z_i^*)} u_i(x_{i0}^*, y)$, whose embedding, $x_i^* : \omega \in \{0\} \cup \Omega_i \mapsto x_{i\omega}^*$, defined from above, is obviously a consumption plan, $x_i^* \in X_i$. □

Assertion *(viii)* Let $i \in I$ be given and $x_i^* \in X_i$ be defined from above. Let $\varphi_i : (x,\omega,z) \in \mathbb{R}_+^H \times \Omega_i \times \mathbb{R}^J \mapsto \arg\max_{y \in B_i(\omega,z)} \; u_i(x,y)$ be defined on its domain. By the same token as above, φ_i and $U_i : (x,\omega,z) \in \mathbb{R}_+^H \times \Omega_i \times \mathbb{R}^J \mapsto u_i(x, \varphi_i(x,\omega,z))$ are continuous mappings and, moreover, the relations $u_i(x_{i0}^*, x_{i\omega}^*) = U_i(x_{i0}^*, \omega, z_i^*)$ and $u_i(x_{i0}^n, x_{i\Phi_i^n(\omega)}^n) = U_i(x_{i0}^n, \Phi_i^n(\omega), z_i^n)$ hold, for every $(\omega, n) \in \Omega_i \times \mathbb{N}$. Then, the uniform continuity of u_i and U_i on compact sets, yield, from above:

$$(I) \qquad \forall \varepsilon > 0, \; \exists N_\varepsilon \in \mathbb{N} : \forall n > N_\varepsilon, \; \forall \omega \in \Omega_i, \; | \; u_i(x_{i0}^*, x_{i\omega}^*) - u_i(x_{i0}^n, x_{i\Phi_i^n(\omega)}^n) \; | \; < \varepsilon.$$

Moreover, we recall the following definitions, for every $n > N$:

(II) $\quad U_i^{\pi_i}(x_i^*) := \int_{\omega\in\Omega_i} u_i(x_{i0}^*, x_{i\omega}^*)d\pi_i(\omega);$

(III) $\quad u_i^n(x_i^n) := \sum_{s\in\underline{\mathbf{S}}} \frac{u_i(x_{io}^n, x_{is}^n)}{2^{n+1}\#\underline{\mathbf{S}}} + (1- \frac{1}{2^{n+1}}) \sum_{\omega\in\Omega_i^n} u_i(x_{i0}^n, x_{i\omega}^n)\pi_i^n(\omega).$

Then, Lemma 3-(vii) results immediately from relations (I)-(II)-(III) above. □

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